Criticality and Covered Area Fraction in Confetti Percolation

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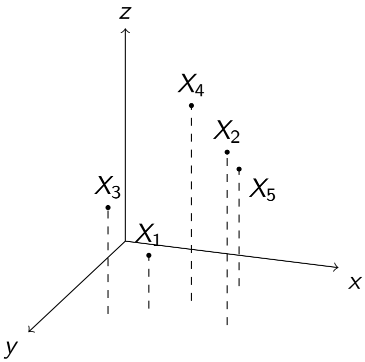
(This talk is a joint work with Rahul Roy)

17 July, 2025

Confetti Percolation



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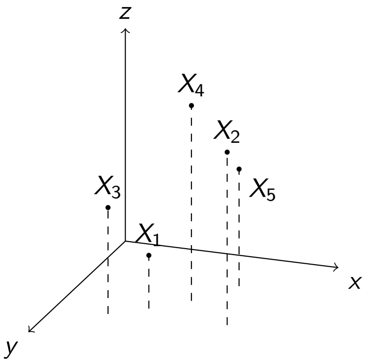

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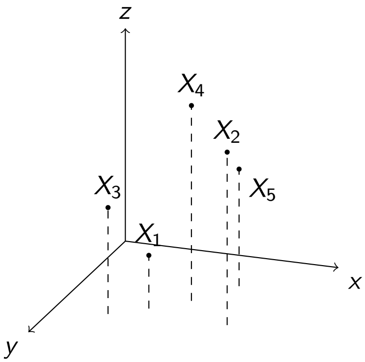

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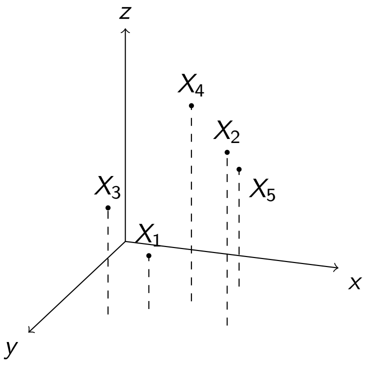

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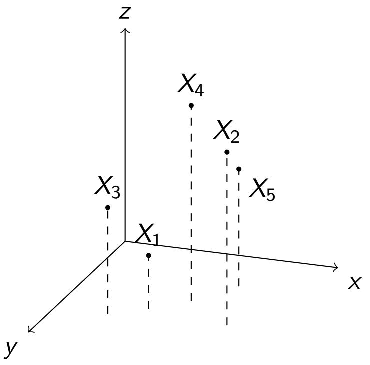

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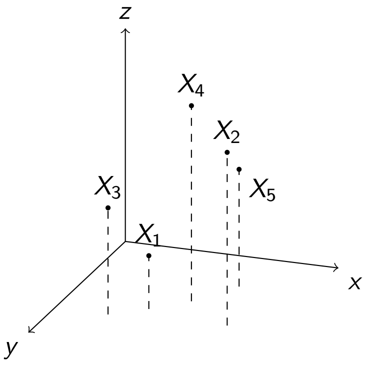

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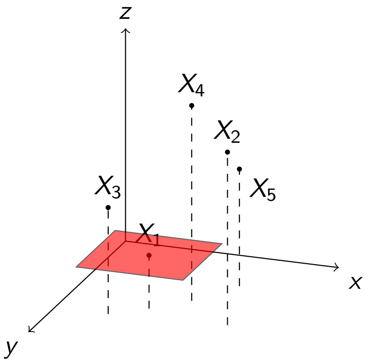

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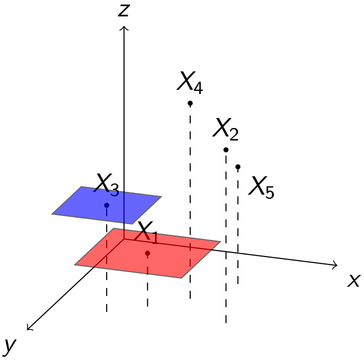

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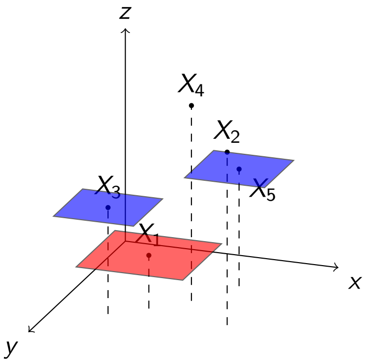

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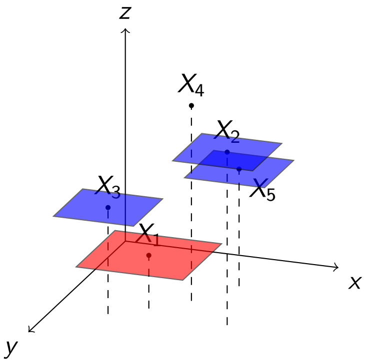

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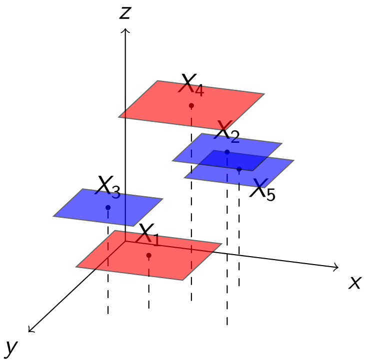

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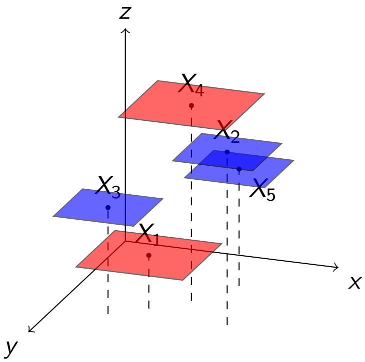

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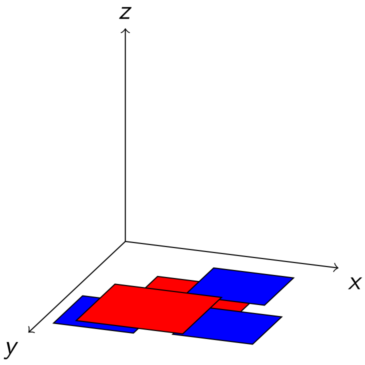

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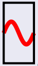
Proposition 1 (Exponential Decay Property).

Consider the confetti model $(\mathcal{P}, \lambda, \rho, \beta)$. There exists a constant $\kappa_0 \in (0, 1)$ such that whenever $\mathbb{P}_\lambda\Big($  $_{N\times 3N}\Big) < \kappa_0$ for some $N \geq R$, we have

$$\mathbb{P}_\lambda(\mathrm{diam}(C^{\mathrm{red}}(\mathbf{0})) \geq a) \leq c_1 \exp(-c_2 a)$$

for all $a > 0$ and for some positive constants c_1 and c_2 depending only on κ_0.

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In particular, when $\rho \overset{d}{=} \beta$, due to symmetry we have

$$\lambda_c(\rho, \beta) = 1/2.$$

Sharp Phase Transition

- We define

$$\theta_n(\lambda) := \mathbb{P}_\lambda(\mathrm{diam}(C^{\mathrm{red}}(\mathbf{0})) \geq n)$$

and

$$\theta(\lambda) := \mathbb{P}_\lambda(\mathrm{diam}(C^{\mathrm{red}}(\mathbf{0})) = \infty).$$

Proposition 2 (Sharp Phase Transition).

For any $\lambda < \lambda_c(\rho, \beta)$, there exists $c_\lambda > 0$ such that for any $n \geq 1$,

$$\theta_n(\lambda) \leq \exp\left(-c_\lambda n\right).$$

Furthermore, there exists $c > 0$ such that for any $\lambda_c(\rho, \beta) < \lambda$,

$$\theta(\lambda) \geq c(\lambda - \lambda_c(\rho, \beta)).$$

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- **Question:** What is value of $\lambda_c(\rho, \beta)$ in general?

Covered Area Fraction

- We define

$$\text{CAF}_{\text{red}}(\lambda, \rho, \beta) := \mathbb{E}_\lambda \left[\ell(\text{red region in } [0,1]^2) \right],$$

and

$$\text{CAF}_{\text{blue}}(\lambda, \rho, \beta) := \mathbb{E}_\lambda \left[\ell(\text{blue region in } [0,1]^2) \right],$$

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Theorem 3.

For any $t \in (0,1)$, there exists a confetti model $(\mathcal{P}, \lambda, \rho, \beta)$, with ρ and β random, for which $\mathrm{CAF}_{\mathrm{red}}(\lambda, \rho, \beta) > t$ but red does not percolate.

An Equivalent Construction of the Confetti Model

- Let $\mathcal{R}$ and $\mathcal{B}$ be two independent Poisson point processes on $\mathbb{R}^2 \times (0, \infty)$ of intensities λ_r and λ_b respectively.
- At points of $\mathcal{R}$ we place red squares of side length ρ and at points of $\mathcal{B}$ we place blue squares of side length β.
- We denote this model by $(\mathcal{R}, \lambda_r, \rho; \mathcal{B}, \lambda_b, \beta)$ and we note that, on scaling this model is equivalent to the model $(\mathcal{P}, \frac{\lambda_r}{\lambda_r+\lambda_b}, \rho, \beta)$.

Transitivity Condition

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Let ρ, β and γ be three positive fixed **constants** (not necessarily equal), and let λ_r, λ_b and λ_g be the intensities of the three independent Poisson point processes red, blue and green, labeled $\mathcal{R}$, $\mathcal{B}$ and $\mathcal{G}$, respectively. Suppose red is supercritical in the red/blue confetti model $(\mathcal{R}, \lambda_r, \rho; \mathcal{B}, \lambda_b, \beta)$ and blue is supercritical in the blue/green confetti model $(\mathcal{B}, \lambda_b, \beta; \mathcal{G}, \lambda_g, \gamma)$, then red is supercritical in the red/green confetti model $(\mathcal{R}, \lambda_r, \rho; \mathcal{G}, \lambda_g, \gamma)$.

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- The transitivity condition is equivalent to

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for any ρ, β and γ constants, not necessarily equal.

Transitivity Condition

Theorem 4.

The following are equivalent:

(i) The transitivity condition holds.

(ii) For any ρ, β positive constants the critical covered area fraction for $(\mathcal{P}, \lambda, \rho, \beta)$ equals $1/2$.

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- We believe that the transitivity condition holds and simulations suggest the veracity of our belief. More details, the source code etc. of the simulation are available at https://www.isid.ac.in/~rahul/index.php/source-code/

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- This implies (ii) $\Leftrightarrow$ (iii).

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- Suppose the transitivity condition holds, and if possible, suppose we have a confetti model $(\mathcal{R}, \lambda_r, \rho; \mathcal{B}, \lambda_b, \beta)$ with $\text{CAF}_{\text{red}}(\lambda, \rho, \beta) < 1/2$, i.e., $\lambda_r \rho^2 < \lambda_b \beta^2$, but red is supercritical.

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Proposition 4.

For any $\sigma > 1$, there exists $0 < \beta_1 < \beta_2 < \infty$, such that for all $\beta \notin (\beta_1, \beta_2)$, blue is supercritical in the model $(\mathcal{R}, 1, 1; \mathcal{B}, \frac{\sigma}{\beta^2}, \beta)$.

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- Hypothesis: Transitivity condition holds, red is supercritical in $(\mathcal{R}, 1, 1; \mathcal{B}, \mu^k, \nu^k)$ for any $k \in \mathbb{N}$, and $\mu\nu^2 > 1$. ($\nu = 1$ is not possible)

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- Choosing k sufficiently large we can get $\mu^k\nu^{2k} > \sigma$ and $\nu^k \notin (\beta_1, \beta_2)$. So, red can not percolate in $(\mathcal{R}, 1, 1; \mathcal{B}, \frac{\mu^k\nu^{2k}}{\nu^{2k}}, \nu^k)$. (Contradiction!!!)

Proof of Theorem 4. (i) $\Rightarrow$ (ii)

- Hypothesis: Transitivity condition holds, red is supercritical in $(\mathcal{R}, 1, 1; \mathcal{B}, \mu^k, \nu^k)$ for any $k \in \mathbb{N}$, and $\mu\nu^2 > 1$. ($\nu = 1$ is not possible)

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- This completes the proof.

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- To prove transitivity, it is enough to show that $\mathbb{P}(C^{\text{red}}(\mathbf{0}) \cap S_n \neq \emptyset)$ is
  (a) monotonic (either non-increasing or non-decreasing) for $\beta \in (0, 1)$ and
  (b) monotonic (either non-increasing or non-decreasing) for $\beta \in (1, \infty)$ for each $n \geq 1$.

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  - (b) monotonic (either non-increasing or non-decreasing) for $\beta \in (1, \infty)$ for each $n \geq 1$.
- By Russo's formula, for $A := \{C^{\text{red}}(\mathbf{0}) \cap S_n \neq \emptyset\}$,

$$\begin{aligned}
&\mathbb{P}_{\beta+\delta}(A) - \mathbb{P}_{\beta}(A) \\
&= \left(\frac{\sigma}{\sigma + \beta^2} - \frac{\sigma}{\sigma + (\beta+\delta)^2} \right) \cdot \mathbb{E}_{\beta}(\#\ \text{'colour pivotal' points for } A) \\
&\quad - \frac{\sigma}{\sigma + (\beta+\delta)^2} \cdot \mathbb{E}_{\beta}(\#\ \text{'size pivotal' points for } A).
\end{aligned}$$

- Here, the i-th Poisson point is 'colour pivotal' for A indicates if we change its colour from red to blue, it will affect the occurrence of A. Similarly, the i-th Poisson point is 'size pivotal' for A means if the point is blue and we change the side length of the associated blue square from β to $\beta + \delta$, it will affect the occurrence of A.

Thank You