Extremal Process of Last Progeny Modified Branching Random Walks

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(This talk is a joint work with Bastien Mallein)

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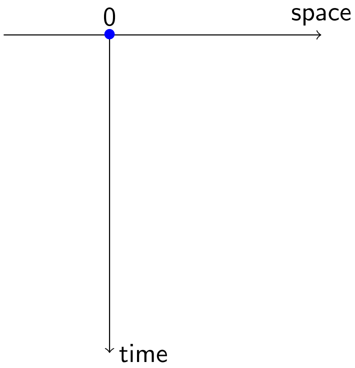

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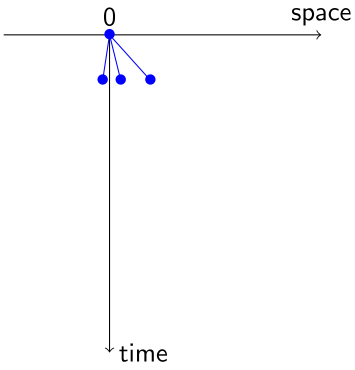

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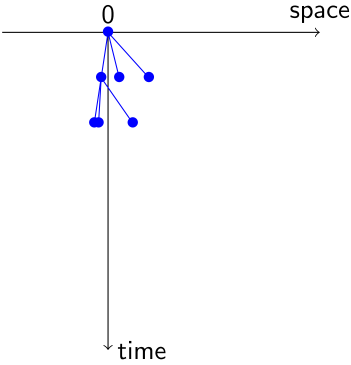

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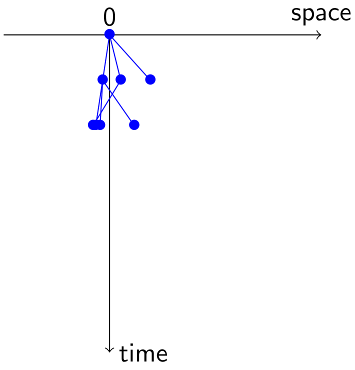

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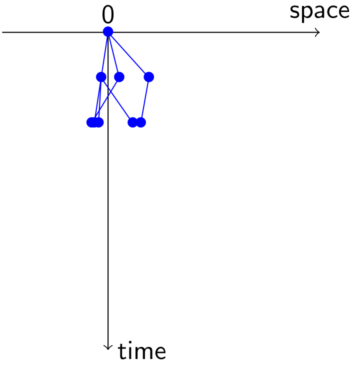

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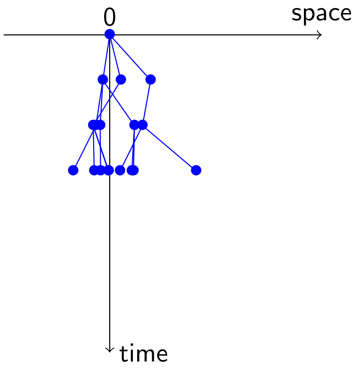

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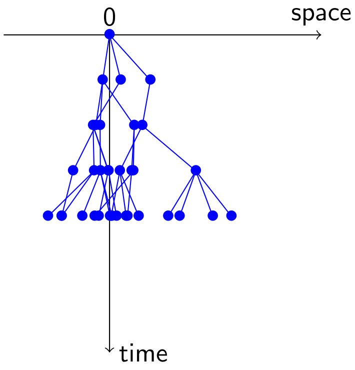

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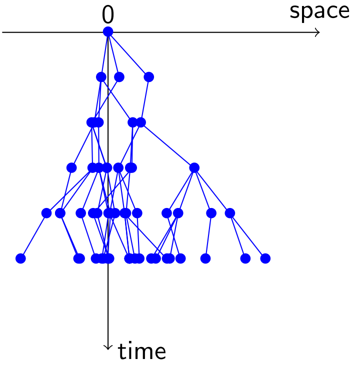

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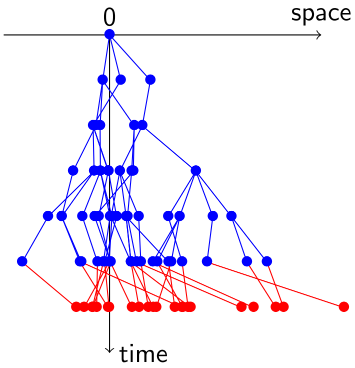

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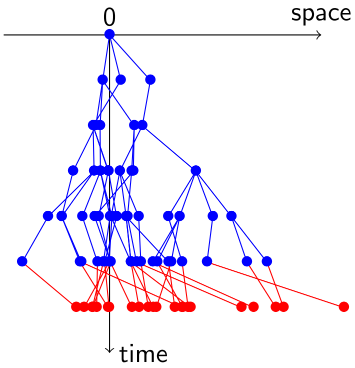

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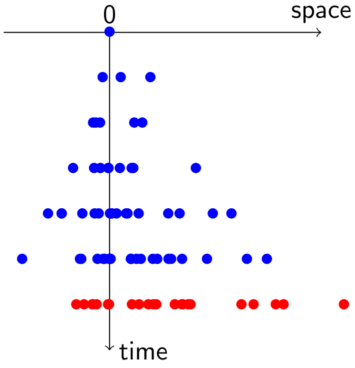

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Notation

- *Empirical measure of the classical BRW* $\mathcal{X}_n := \sum_{|v|=n} \delta_{S_v}$.
- *Empirical measure of the LPM-BRW* $\mathcal{E}_n := \sum_{|v|=n} \delta_{S_v+Y_v}$.

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Extremal Process of Classical BRW

- Branching Random Walk was first introduced by Hammersley [1974]. After that, after many breakthroughs by many probabilists, Madaule in 2017 showed that under mild conditions, for

$$m_n = n\frac{\kappa(\theta_0)}{\theta_0} - \frac{3}{2\theta_0}\log n, \qquad \tau_{-m_n}\mathcal{X}_n \xrightarrow{d} \mathrm{DPPP}(c\theta_0 Z_\infty e^{-\theta_0 x}\,\mathrm{d}x).$$

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- In this work, we aim to get similar asymptotics for the *extremal process of the modified branching random walk*.

Notations

- For a point process $\mathcal{Z} = \sum_{j\geq 1} \delta_{\xi_j}$, we write

$$m(t) := \mathbb{E}\left[\int_{\mathbb{R}} e^{tx}\,\mathcal{Z}(dx)\right] = \mathbb{E}\left[\sum_{j\geq 1} e^{t\xi_j}\right].$$

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- Three cases to be considered:
  - **Below the Boundary Case:** $\theta < \theta_0 \leq \infty$;
  - **Boundary Case:** $\theta = \theta_0 < \infty$; and
  - **Above the Boundary Case:** $\theta_0 < \theta < \infty$.

Below the Boundary Case : $\theta < \theta_0 \leq \infty$

Theorem 1. [G. and Mallein (2021)]

Let $\theta > 0$ such that $\kappa(\theta) < \infty$. We assume that $W_n(\theta) := \sum_{|u|=n} e^{\theta S_u - n\kappa(\theta)}$ are uniformly integrable and there exists a constant $L \in (0, \infty)$ satisfying

$$\mathbb{P}(Y > x) \sim Le^{-\theta x} \quad \text{as } x \to \infty.$$

Then, writing

$$m_n = n\frac{\kappa(\theta)}{\theta} + \frac{1}{\theta}\log L,$$

the extremal process $\tau_{-m_n}\mathcal{E}_n$ converges in law to a $\mathrm{PPP}(\theta W_\infty(\theta)e^{-\theta x}\,\mathrm{d}x)$.

Below the Boundary Case : $\theta < \theta_0 \leq \infty$

Theorem 2. [G. and Mallein (2021)]

Let $\theta > 0$ such that $\kappa(\theta) < \infty$. We assume that $W_n(\theta) := \sum_{|u|=n} e^{\theta S_u - n\kappa(\theta)}$ are uniformly integrable and there exists $\delta > 0$ such that $\kappa(\theta+\delta) + \kappa(\theta-\delta) < \infty$ and there exists a regularly varying function L at ∞ with index α satisfying

$$\mathbb{P}(Y > x) \sim L(x) e^{-\theta x} \quad \text{as } x \to \infty.$$

Then, writing

$$m_n = n\frac{\kappa(\theta)}{\theta} + \frac{1}{\theta}\log L(n) \quad \text{and} \quad c_1 = \left(\frac{\kappa(\theta)}{\theta} - \kappa'(\theta)\right)^{\alpha},$$

the extremal process $\tau_{-m_n}\mathcal{E}_n$ converges in law to a $\mathrm{PPP}(c_1 \theta W_\infty(\theta) e^{-\theta x}\, \mathrm{d}x)$.

Boundary Case : $\theta = \theta_0 < \infty$

Theorem 3. [G. and Mallein (2021)]

We assume that $\kappa''(\theta_0) < \infty$ and $\mathbb{E}\left[W_1(\theta_0)(\log_+ W_1(\theta_0))^2\right] + \mathbb{E}[\bar{W}_1 \log_+(\bar{W}_1)] < \infty$, where $\bar{W}_1 = \sum_{|u|=1}(\kappa'(\theta_0) - S_u)_+ e^{\theta_0 S_u - \kappa(\theta_0)}$ and $x_+ = \max(x, 0)$. We also assume that there exists a regularly varying function L at ∞ with index $\alpha \in (-2, 0)$ satisfying

$$\mathbb{P}(Y > x) \sim L(x) e^{-\theta_0 x} \quad \text{as } x \to \infty.$$

Then, writing

$$m_n = n\frac{\kappa(\theta_0)}{\theta_0} + \frac{1}{\theta_0} \log L\left(\sqrt{n}\right) - \frac{1}{2\theta_0} \log n$$

$$\text{and} \quad c_2 = \sqrt{\frac{2}{\pi \kappa''(\theta_0)}} \left(2\kappa''(\theta_0)\right)^{\frac{\alpha}{2}} \Gamma\left(\frac{\alpha}{2} + 1\right),$$

the extremal process $\tau_{-m_n}\mathcal{E}_n$ converges in law to a $\mathrm{PPP}(c_2 \theta_0 Z_\infty e^{-\theta_0 x}\,\mathrm{d}x)$.

Here, $Z_\infty \overset{a.s.}{=\!=} \lim_{n\to\infty} \sum_{|u|=n}(n\kappa'(\theta_0) - S_u) e^{\theta_0 S_u - n\kappa(\theta_0)}$.

Boundary Case : $\theta = \theta_0 < \infty$

Theorem [Madaule (2017)]

Under mild conditions, for

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Above the Boundary Case : $\theta_0 < \theta < \infty$

Theorem 4. [G. and Mallein (2021)]

We assume that the reproduction law of the BRW is non-lattice, $\kappa''(\theta_0) < \infty$ and that $\mathbb{E}\left[W_1(\theta_0)(\log_+ W_1(\theta_0))^2\right] + \mathbb{E}[\bar{W}_1 \log_+(\bar{W}_1)] < \infty$, where $\bar{W}_1 = \sum_{|u|=1}(\kappa'(\theta_0) - S_u)_+ e^{\theta_0 S_u - \kappa(\theta_0)}$ and $x_+ = \max(x, 0)$. We also assume that there exist $C > 0$ and $\theta > \theta_0$ satisfying

$$\mathbb{P}(Y > x) \leq Ce^{-\theta x} \text{ for all } x \in \mathbb{R}.$$

Then, writing

$$m_n = n\frac{\kappa(\theta_0)}{\theta_0} - \frac{3}{2\theta_0}\log n,$$

the extremal process $\tau_{-m_n}\mathcal{E}_n$ converges in law to

$$\sum_{i\in\mathbb{N}} \delta_{z_i + Y_i},$$

where $(z_i, i \in \mathbb{N})$ are the atoms of the limiting extremal process of the BRW and (Y_i) are i.i.d. copies of Y.

Proof of Theorem 4

Need to show : For any non-negative continuous compactly supported function φ,

$$\mathbb{E}\left[e^{-\sum_{|u|=n}\varphi(S_u+Y_u-m_n)}\right] \to \mathbb{E}\left[e^{-\sum_{i\geq 1}\varphi(z_i+Y_i)}\right].$$

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Let χ_K be a continuous function such that $\mathbb{1}_{[-K,K]}\leq\chi_K\leq\mathbb{1}_{[-2K,2K]}$.

$$\begin{aligned}
&\left|\mathbb{E}\left[e^{-\sum_{|u|=n}g_\varphi(S_u-m_n)}\right]-\mathbb{E}\left[e^{-\sum_{i\geq 1}g_\varphi(z_i)}\right]\right|\\
&\leq\left|\mathbb{E}\left[e^{-\sum_{|u|=n}g_\varphi(S_u-m_n)}\right]-\mathbb{E}\left[e^{-\sum_{|u|=n}\chi_K g_\varphi(S_u-m_n)}\right]\right|\\
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&\quad + \left|\mathbb{E}\left[e^{-\sum_{|u|=n}\chi_K g_\varphi(S_u-m_n)}\right] - \mathbb{E}\left[e^{-\sum_{i\geq 1}\chi_K g_\varphi(z_i)}\right]\right| \xrightarrow{Madaule(2017)} 0 \\
&\quad + \left|\mathbb{E}\left[e^{-\sum_{i\geq 1}\chi_K g_\varphi(z_i)}\right] - \mathbb{E}\left[e^{-\sum_{i\geq 1}g_\varphi(z_i)}\right]\right|
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$$\mathbb{E}\left[e^{-\sum_{|u|=n}\varphi(S_u+Y_u-m_n)}\right] \to \mathbb{E}\left[e^{-\sum_{i\geq 1}\varphi(z_i+Y_i)}\right].$$

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Let χ_K be a continuous function such that $\mathbb{1}_{[-K,K]} \leq \chi_K \leq \mathbb{1}_{[-2K,2K]}$.

$$\begin{aligned}
&\left|\mathbb{E}\left[e^{-\sum_{|u|=n}g_\varphi(S_u-m_n)}\right] - \mathbb{E}\left[e^{-\sum_{i\geq 1}g_\varphi(z_i)}\right]\right| \\
&\leq \left|\mathbb{E}\left[e^{-\sum_{|u|=n}g_\varphi(S_u-m_n)}\right] - \mathbb{E}\left[e^{-\sum_{|u|=n}\chi_K g_\varphi(S_u-m_n)}\right]\right| \\
&\quad + \left|\mathbb{E}\left[e^{-\sum_{|u|=n}\chi_K g_\varphi(S_u-m_n)}\right] - \mathbb{E}\left[e^{-\sum_{i\geq 1}\chi_K g_\varphi(z_i)}\right]\right| \xrightarrow{Madaule(2017)} 0 \\
&\quad + \left|\mathbb{E}\left[e^{-\sum_{i\geq 1}\chi_K g_\varphi(z_i)}\right] - \mathbb{E}\left[e^{-\sum_{i\geq 1}g_\varphi(z_i)}\right]\right| \xrightarrow{DCT} 0
\end{aligned}$$

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$$\left|\mathbb{E}\left[e^{-\sum_{|u|=n} g_\varphi(S_u-m_n)}\right] - \mathbb{E}\left[e^{-\sum_{i\geq 1} g_\varphi(z_i)}\right]\right|$$

$$\leq \left|\mathbb{E}\left[e^{-\sum_{|u|=n} g_\varphi(S_u-m_n)}\right] - \mathbb{E}\left[e^{-\sum_{|u|=n} \chi_K g_\varphi(S_u-m_n)}\right]\right| = I(n,K) \text{ (say)}$$

$$+ \left|\mathbb{E}\left[e^{-\sum_{|u|=n} \chi_K g_\varphi(S_u-m_n)}\right] - \mathbb{E}\left[e^{-\sum_{i\geq 1} \chi_K g_\varphi(z_i)}\right]\right| \xrightarrow{Madaule(2017)} 0$$

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Proof of Theorem 4

$$I(n,K) \leq \mathbb{E}\left[\left(\sum_{|u|=n}(1-\chi_K)g_\varphi(S_u-m_n)\right)\wedge 1\right] \qquad \text{(using } |e^{-a}-e^{-b}| \leq |a-b| \wedge 1\text{)}$$

Proof of Theorem 4

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$$\leq \mathbb{E}\Bigg[\Bigg(\sum_{|u|=n} g_\varphi(S_u - m_n)\mathbb{1}_{S_u - m_n \geq K}\Bigg) \wedge 1\Bigg]$$

$$+ \mathbb{E}\Bigg[\Bigg(\sum_{|u|=n} g_\varphi(S_u - m_n)\mathbb{1}_{S_u - m_n \leq -K}\Bigg) \wedge 1\Bigg]$$

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Since $\mathbb{P}(Y>x)\le Ce^{-\theta x}$, we have $g_\varphi(x)\le C'e^{\theta x}$, for some $C'>0$.

Proof of Theorem 4

$$I(n,K) \le \mathbb{E}\left[\left(\sum_{|u|=n}(1-\chi_K)g_\varphi(S_u - m_n)\right)\wedge 1\right] \qquad \text{(using } |e^{-a}-e^{-b}| \le |a-b|\wedge 1\text{)}$$

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$$J(n,K) \le \mathbb{E}\left[\left(C' \sum_{|u|=n} e^{\theta(S_u - m_n)}\mathbb{1}_{S_u - m_n \le -K}\right)\wedge 1\right]$$

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Proof of Theorem 4

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I(n,K) &\le \mathbb{E}\left[\left(\sum_{|u|=n}(1-\chi_K)g_\varphi(S_u-m_n)\right)\wedge 1\right] \qquad \text{(using } |e^{-a}-e^{-b}|\le |a-b|\wedge 1\text{)}\\
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$$+\,\mathbb{P}\left(\sum_{|u|=n} e^{\theta_1(S_u - m_n)} \geq K\right) \xrightarrow{Madaule(2017)} 0. \text{ (proved)}$$

Below the Boundary Case : $\theta < \theta_0 \leq \infty$

Theorem 1. [G. and Mallein (2021)]

Let $\theta > 0$ such that $\kappa(\theta) < \infty$. We assume that $W_n(\theta) := \sum_{|u|=n} e^{\theta S_u - n\kappa(\theta)}$ are uniformly integrable and there exists a constant $L \in (0, \infty)$ satisfying

$$\mathbb{P}(Y > x) \sim L e^{-\theta x} \quad \text{as } x \to \infty.$$

Then, writing

$$m_n = n\frac{\kappa(\theta)}{\theta} + \frac{1}{\theta}\log L,$$

the extremal process $\tau_{-m_n}\mathcal{E}_n$ converges in law to a $\mathrm{PPP}(\theta W_\infty(\theta) e^{-\theta x}\,\mathrm{d}x)$.

Proof of Theorem 1

Need to show : For any non-negative continuous compactly supported function φ,

$$\mathbb{E}\left[e^{-\sum_{|u|=n}\varphi(S_u+Y_u-m_n)}\right] \to \mathbb{E}\left[e^{-W_\infty(\theta)c_\varphi(\theta)}\right],\text{ where } c_\varphi(\theta) = \int \theta e^{-\theta z}(1-e^{-\varphi(z)})\,\mathrm{d}z.$$

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We define a function g_φ as

$$e^{-g_\varphi(x)} := \mathbb{E}[e^{-\varphi(x+Y)}].$$

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We show that $g_\varphi(x) \sim Le^{\theta x}c_\varphi(\theta)$ as $x \to -\infty$

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We show that $\left|g_\varphi(x) - Le^{\theta x}c_\varphi(\theta)\right| < \epsilon e^{\theta x}$ for all $x \le -A$.

Proof of Theorem 1

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Now,

$$\mathbb{E}\left[\left|e^{-\sum_{|u|=n}g_\varphi(S_u-m_n)} - e^{-W_n(\theta)c_\varphi(\theta)}\right|\right]$$

$$\le \mathbb{E}\left[\left|\sum_{|u|=n}g_\varphi(S_u-m_n) - W_n(\theta)c_\varphi(\theta)\right| \wedge 1\right] \quad \text{(since } |e^{-a}-e^{-b}| \le |a-b| \wedge 1 \text{)}$$

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Need to show : For any non-negative continuous compactly supported function φ,

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We show that $\left|g_\varphi(x)-Le^{\theta x}c_\varphi(\theta)\right|<\epsilon e^{\theta x}$ for all $x\leq -A$.

Now,

$$\mathbb{E}\left[\left|e^{-\sum_{|u|=n}g_\varphi(S_u-m_n)}-e^{-W_n(\theta)c_\varphi(\theta)}\right|\right]$$

$$\leq\mathbb{E}\left[\left|\sum_{|u|=n}g_\varphi(S_u-m_n)-\sum_{|u|=n}Le^{\theta(S_u-m_n)}c_\varphi(\theta)\right|\wedge 1\right]\quad(\text{since } e^{\theta m_n}=Le^{n\kappa(\theta)})$$

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We show that $|h_\varphi(x)| < \epsilon e^{\theta x}$ for all $x < -A$, where $h_\varphi(x) := g_\varphi(x) - Le^{\theta x}c_\varphi(\theta)$.

Now,

$$\mathbb{E}\left[\left|e^{-\sum_{|u|=n} g_\varphi(S_u-m_n)} - e^{-W_n(\theta)c_\varphi(\theta)}\right|\right]$$

$$\le \mathbb{E}\left[\left|\sum_{|u|=n}\left(g_\varphi(S_u-m_n) - Le^{\theta(S_u-m_n)}c_\varphi(\theta)\right)\right| \wedge 1\right]$$

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Need to show : For any non-negative continuous compactly supported function φ,

$$\mathbb{E}\left[e^{-\sum_{|u|=n}\varphi(S_u+Y_u-m_n)}\right] \to \mathbb{E}\left[e^{-W_\infty(\theta)c_\varphi(\theta)}\right], \text{ where } c_\varphi(\theta) = \int \theta e^{-\theta z}(1-e^{-\varphi(z)})\,\mathrm{d}z.$$

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Thank You