# Large and Moderate Deviations in Poisson Navigations

**Partha Pratim Ghosh**

Ruhr-Universität Bochum

(This talk is a joint work with B. Jahnel and S. K. Jhawar)

November 10, 2025

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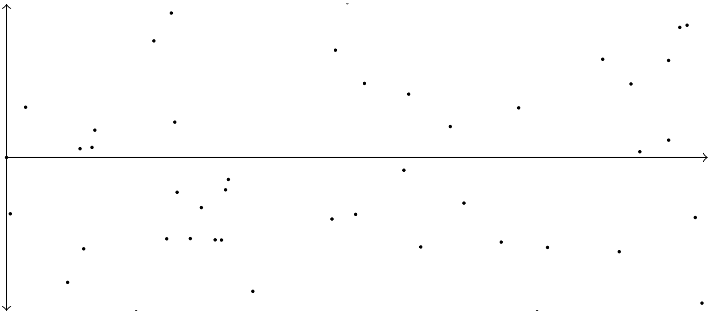

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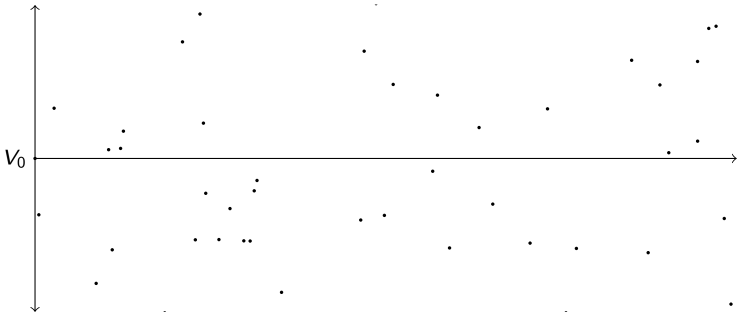

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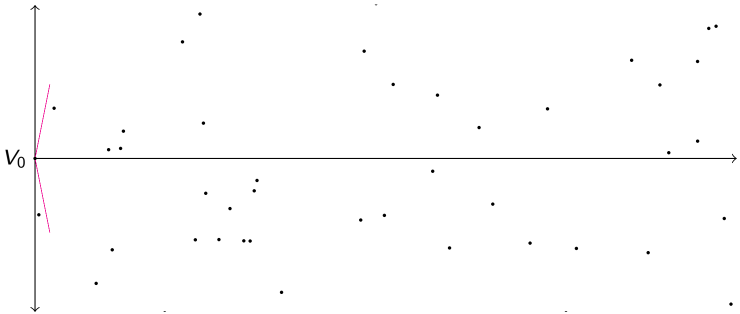

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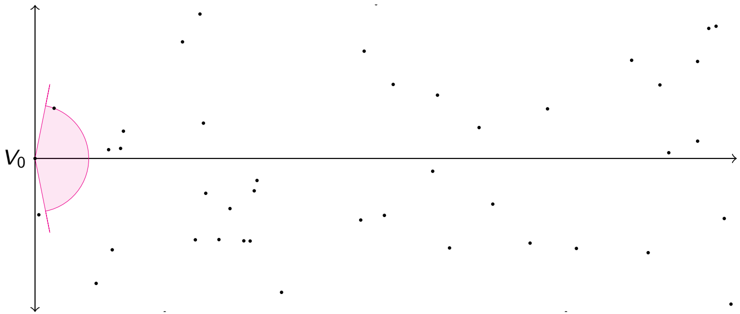

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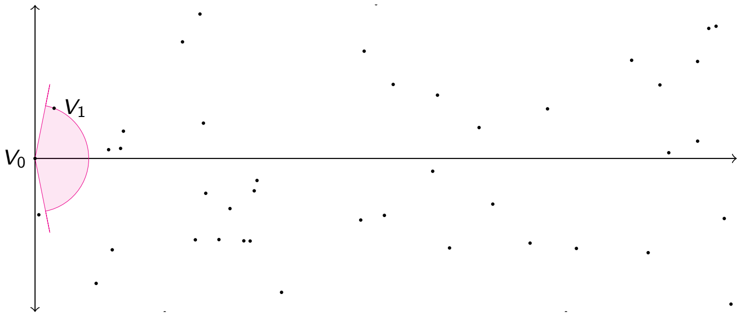

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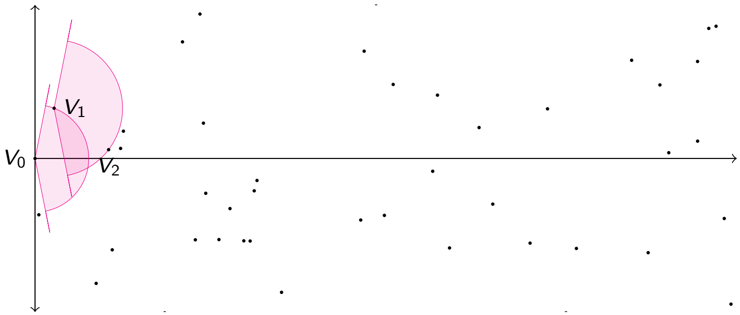

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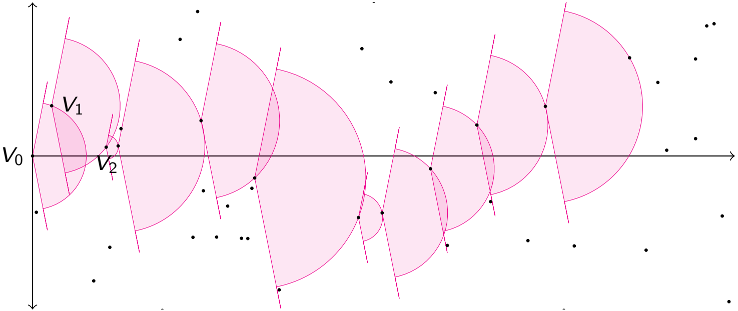

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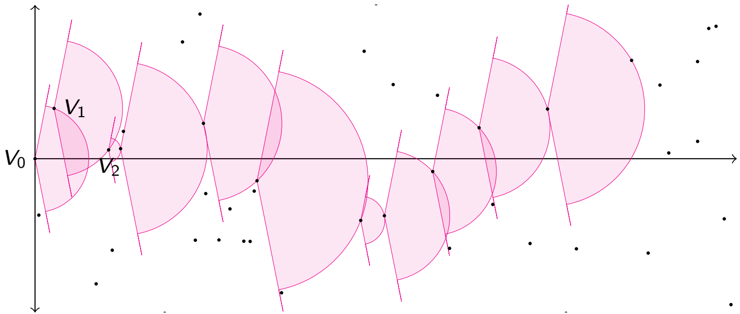
- The node $V_{i+1}$ is called the successor of $V_i \in \mathcal{P}_\lambda$ in the navigation.



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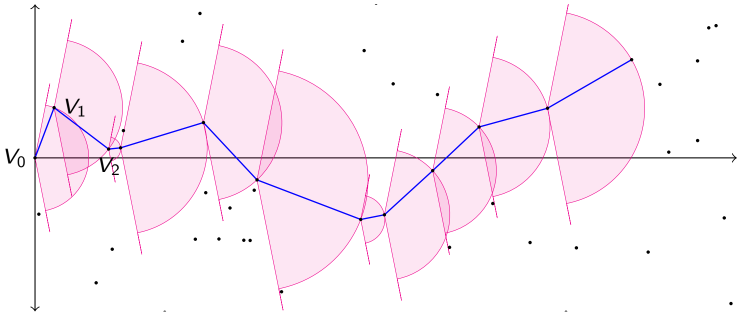

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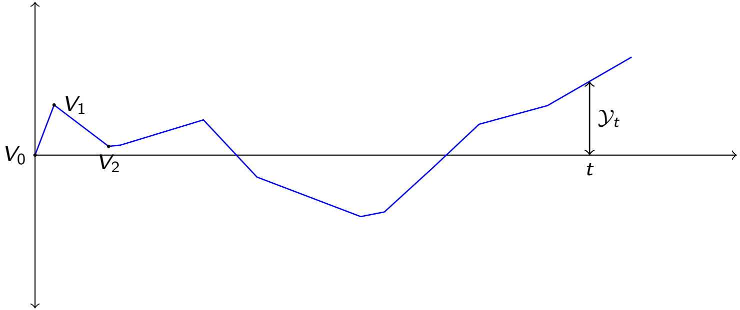

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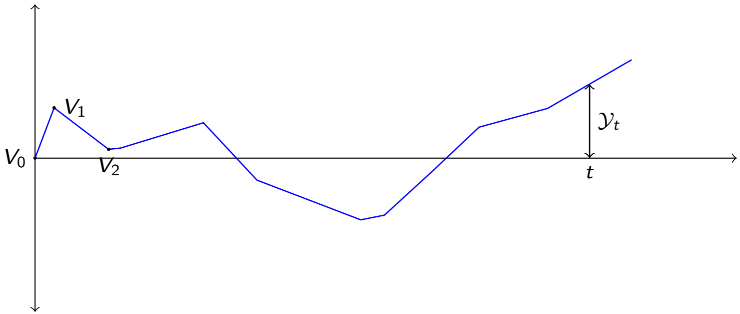

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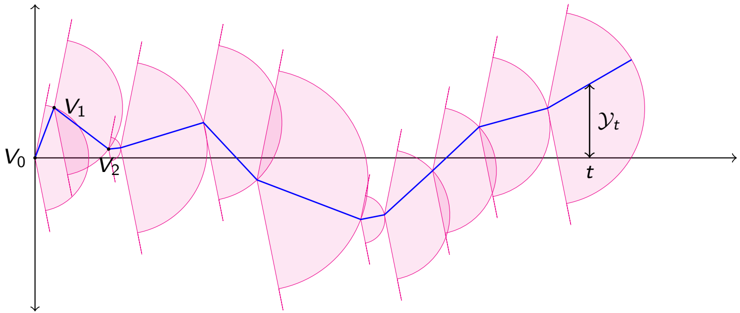

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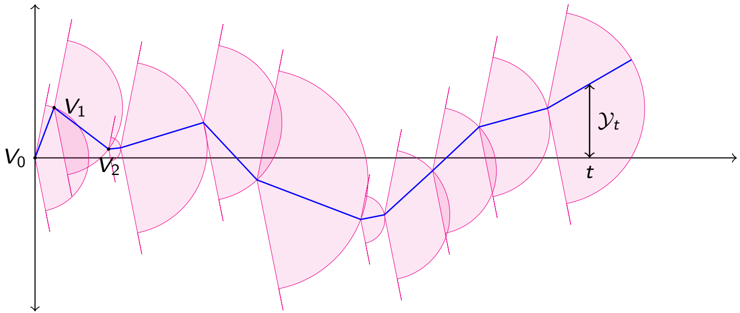

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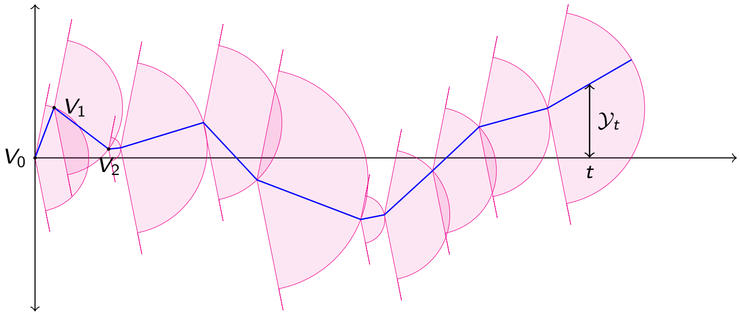
- We need to account for the random horizontal step-sizes, which are not independent of the vertical displacements.



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For any $0 < \lambda$, $0 < \theta \leq \frac{\pi}{2}$, and $0 < \varepsilon < \frac{1}{2}$, the family $\left\{t^{-1/2-\varepsilon}\mathcal{Y}_t\right\}_{t\geq 0}$ obeys the moderate deviation principle with rate $t^{2\varepsilon}$ and rate function $I_{\lambda,\theta}(x) := \rho(\lambda,\theta)x^2$, where $\rho(\lambda,\theta) > 0$. This means that for any Borel set $\Gamma \subseteq \mathbb{R}$,

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- $\beta$ controls the optimal balance between making unusually many steps in order to reach $t$ horizontally, which might be beneficial to reach level $x$ vertically.

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- What about $\theta > \frac{\pi}{4}$? **Challenges:**
  - Dependencies between steps. This can be managed using a renewal structure.
  - The renewal steps have exponential tails.

We have results assuming some control on renewal steps.

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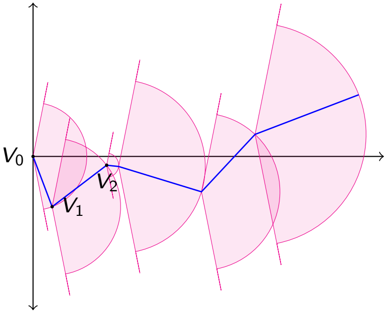


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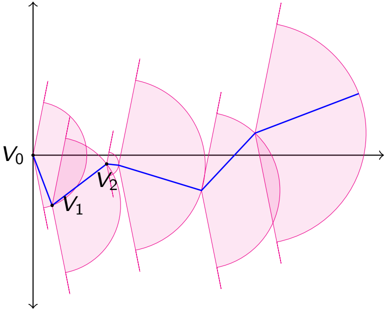


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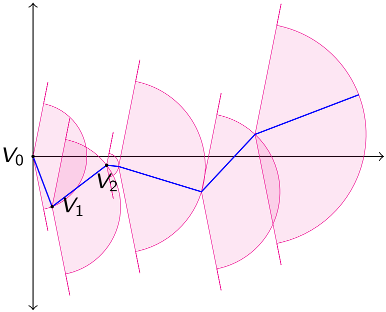


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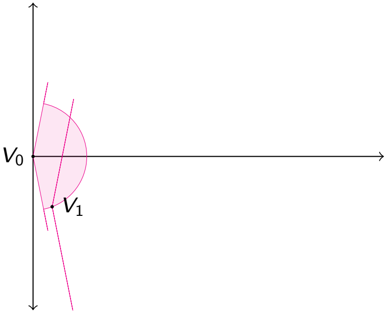


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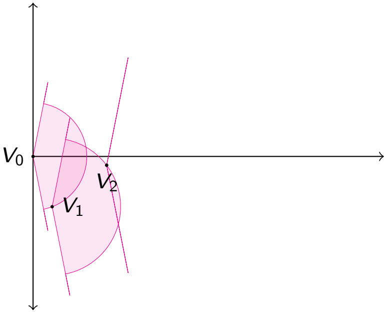


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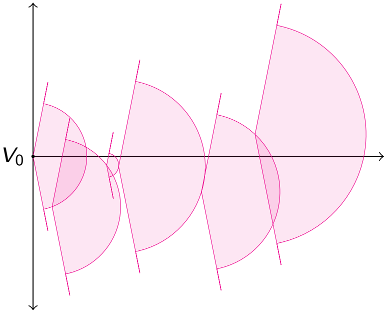


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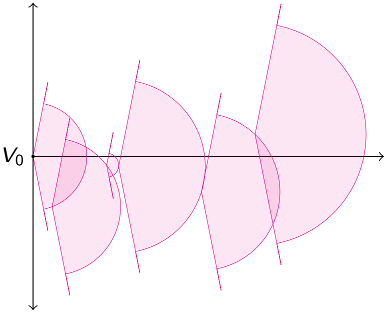


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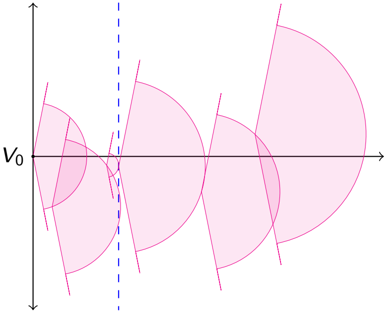


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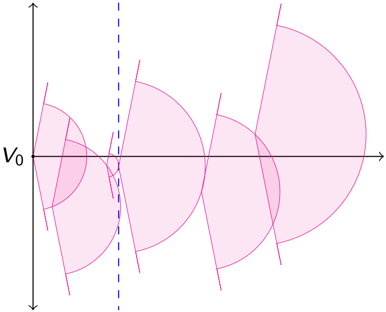


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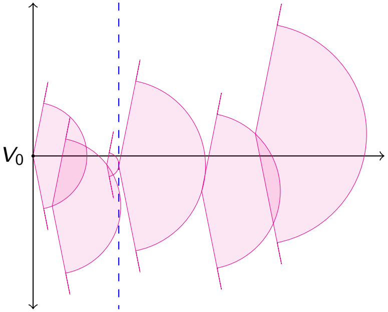
# Strategy of Proof of the MDP for $\frac{\pi}{4} < \theta < \frac{\pi}{2}$

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- Progress at the $i$-th step $U_i := V_i - V_{i-1}$.
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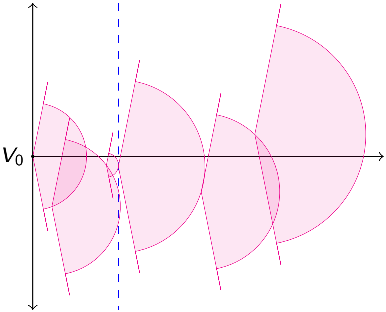


- *"Horizontal Progress"*.

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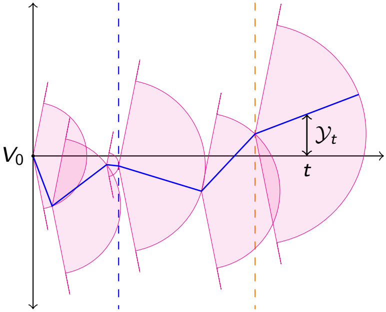


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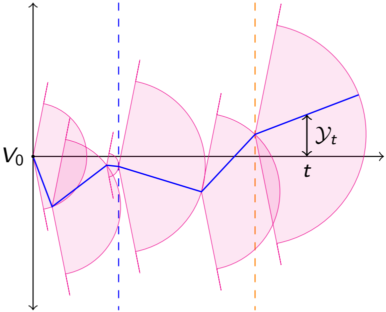


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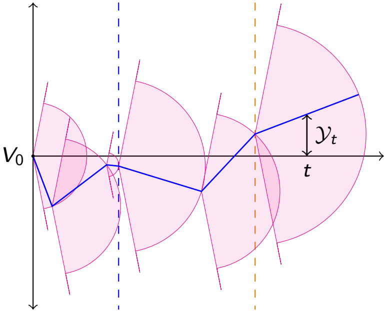


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- Then

$$\boxed{\mathcal{Y}_t = \sum_{i=1}^{K_t'} Y_i' + \sum_{i=\tau_{K_t'}+1}^{K_t} Y_i + (\text{Rest step}),} \quad \text{where} \quad Y_i' = \sum_{j=\tau_{i-1}+1}^{\tau_i} Y_j.$$

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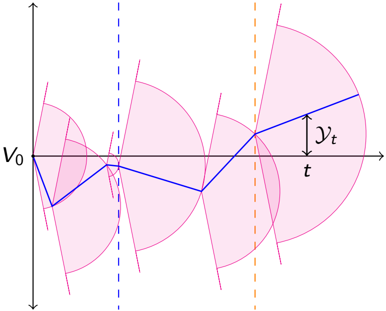


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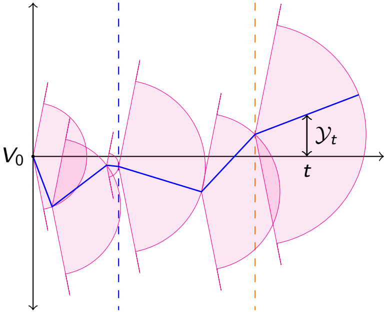


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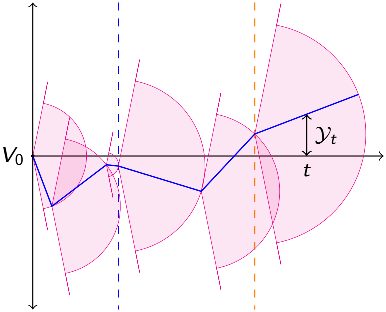


**Step 2: Exponential Equivalence.**

- $\mathcal{Y}_t$ and $\mathcal{Y}'_t$ are exponentially equivalent: $\forall \delta > 0$,

$$\limsup_{t\uparrow\infty} t^{-2\epsilon} \log \mathbb{P}\big(|\mathcal{Y}_t - \mathcal{Y}'_t| \geq \delta\, t^{1/2+\epsilon}\big) = -\infty.$$

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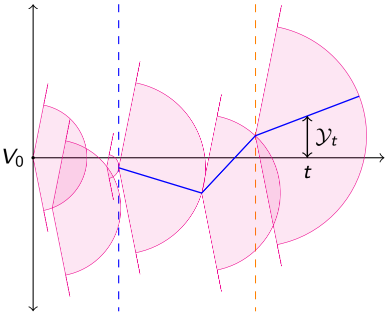


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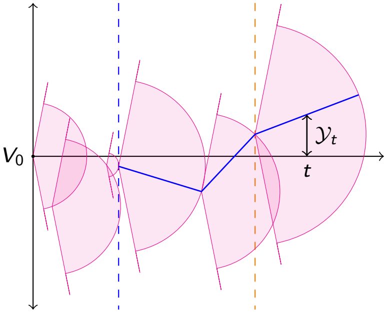


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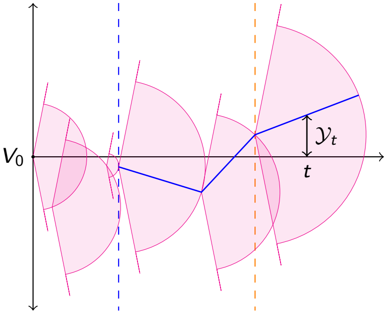


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- $\mathbb{P}\big(|\mathcal{Y}_t - \mathcal{Y}_t'| \geq \delta\, t^{1/2+\epsilon}\big)$

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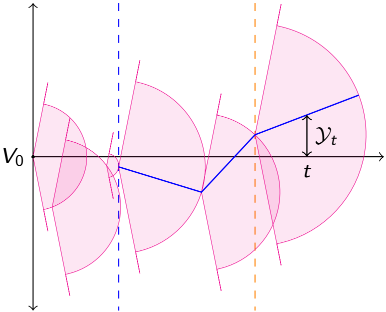


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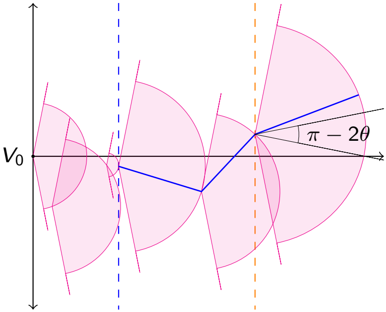


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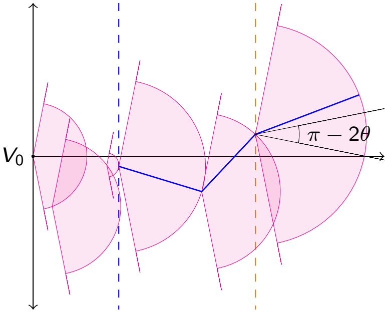


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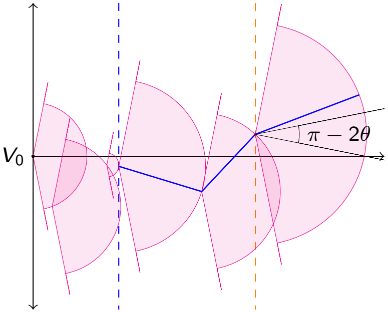


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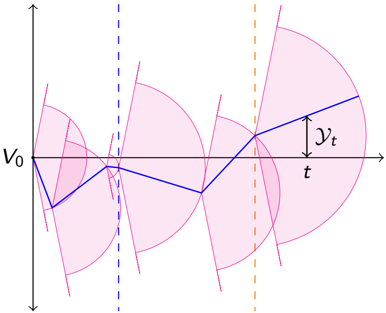


**Step 3: No Randomness in Horizontal Direction.**

- $\mathcal{Y}'_t$ and $\mathcal{Y}^{\varkappa}_t$ are exponentially equivalent: $\forall \delta > 0$,

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where $\varkappa := \mathbb{E}[X'_1]$ and $\mathcal{Y}^{\varkappa}_t = \sum_{i=1}^{\lfloor t/\varkappa \rfloor} Y'_i$.

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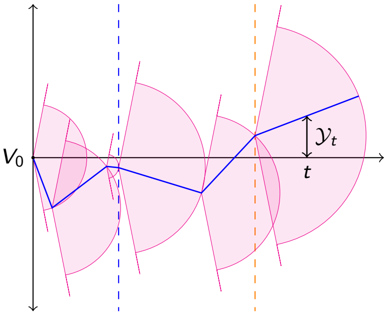


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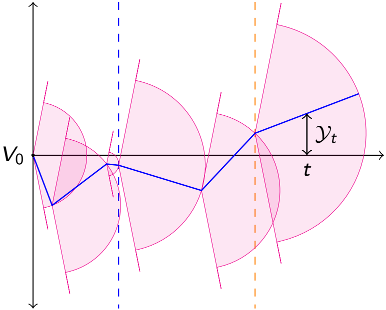
- **Remarks:**
  - Infact, the the process makes the typical number of steps.
  - Crucially uses *exponential* unlikeliness of making unexpected many steps.

# Strategy of Proof of the MDP for $\frac{\pi}{4} < \theta < \frac{\pi}{2}$

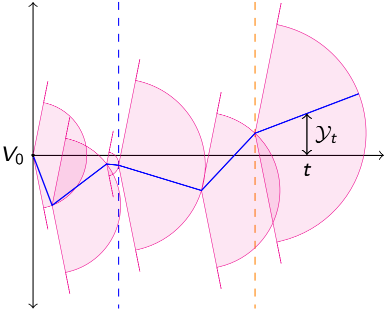



**Step 4: MDP for $\mathcal{Y}_t^{\varkappa}$.**

- $\{t^{-1/2-\varepsilon}\mathcal{Y}_t^{\varkappa}\}_{t\geq 0}$ obeys the moderate-deviation principle with rate $t^{2\varepsilon}$ and rate function

$$I_{\lambda,\theta}^{\varkappa}(x) := \frac{\varkappa \cdot x^2}{2 \cdot \mathbb{E}[Y_1'^2]} = \frac{\mathbb{E}[X_1'] \cdot x^2}{2 \cdot \mathbb{E}[Y_1'^2]}.$$

# Strategy of Proof of the MDP for $\frac{\pi}{4} < \theta < \frac{\pi}{2}$



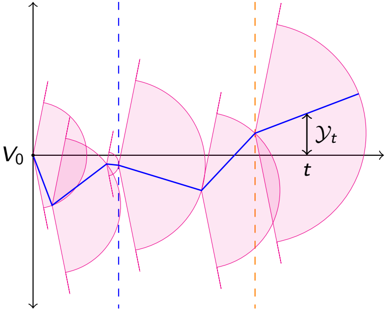

**Step 4: MDP for $\mathcal{Y}_t^{\varkappa}$.**

- $\left\{t^{-1/2-\varepsilon}\mathcal{Y}_t^{\varkappa}\right\}_{t\geq 0}$ obeys the moderate-deviation principle with rate $t^{2\varepsilon}$ and rate function

$$I_{\lambda,\theta}^{\varkappa}(x) := \frac{\varkappa \cdot x^2}{2\cdot \mathbb{E}[Y_1'^2]} = \frac{\mathbb{E}[X_1'] \cdot x^2}{2\cdot \mathbb{E}[Y_1'^2]}.$$

- **Proof:** Application of Eichelsbacher–Löwe criteria. [ESAIM Probab. Stat., 7:209–218.]

# Strategy of Proof of the MDP for $\frac{\pi}{4} < \theta < \frac{\pi}{2}$




**Step 4: MDP for $\mathcal{Y}_t^{\varkappa}$.**

- $\{t^{-1/2-\varepsilon}\mathcal{Y}_t^{\varkappa}\}_{t\geq 0}$ obeys the moderate-deviation principle with rate $t^{2\varepsilon}$ and rate function

$$I_{\lambda,\theta}^{\varkappa}(x) := \frac{\varkappa \cdot x^2}{2 \cdot \mathbb{E}[Y_1'^2]} = \frac{\mathbb{E}[X_1'] \cdot x^2}{2 \cdot \mathbb{E}[Y_1'^2]}.$$

- **Proof:** Application of Eichelsbacher–Löwe criteria. [ESAIM Probab. Stat., 7:209–218.]

(proved)

# References


1. Baccelli, F. and Bordenave, C. (2007). The radial spanning tree of a Poisson point process. *Annals of Applied Probability*, **17**(1):305–359.
2. Bordenave, C. (2008). Navigation on a Poisson point process. *Annals of Applied Probability*, **18**(2):708–746.
3. Coupier, D., Saha, K., Sarkar, A. and Tran, V. C. (2021). The 2d-directed spanning forest converges to the Brownian web. *Annals of Probability*, **49**(1):435–484.
4. Eichelsbacher, P. and Löwe, M. (2003). Moderate deviations for i.i.d. random variables. *ESAIM: Probability and Statistics*, **7**:209–218.
5. **Ghosh, Partha Pratim**, Jahnel, B., and Jhawar, S. K. (2025). Large and moderate deviations in Poisson navigations. *Advances in Applied Probability*.
6. Roy, R., Saha, K. and Sarkar, A. (2016). Random directed forest and the Brownian web. *Annales de l'Institut Henri Poincaré Probabilité et Statistiques*, **52**(3):1106–1143.

**Thank You**