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Right-Most Position of a Last Progeny Modified Branching Random Walk

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Abstract

In this work, we consider a modification of the usual branching random walk (BRW), where we give certain independent and identically distributed (i.i.d.) displacements to all the particles at the n-th generation, which may be different from the driving increment distribution. We call this process *last progeny modified branching random walk (LPM-BRW)*. Depending on the value of a parameter, θ, we classify the model into three distinct cases, namely, the boundary case, below the boundary case, and above the boundary case. Under very minimal assumptions on the underlying point process of the increments, we show that at the *boundary case*, $\theta = \theta_0$, where θ_0 is a parameter value associated with the displacement point process, the maximum displacement converges to a limit after only an appropriate centering, which is of the form $c_1 n - c_2 \log n$. We give an explicit formula for the constants c_1 and c_2 and show that c_1 is exactly the same, while c_2 is $1/3$ of the corresponding constants of the usual BRW [2]. We also characterize the limiting distribution. We further show that below the boundary, $\theta < \theta_0$, the logarithmic correction term is absent. For above the boundary, $\theta > \theta_0$, the logarithmic correction term is exactly the same as that of the classical BRW. For $\theta \leq \theta_0$, we further derive Brunet–Derrida-type results of point process convergence of our LPM-BRW to a Poisson point process. Our proofs are based on a novel method of coupling the maximum displacement with a *linear statistic* associated with a more well-studied process in statistics, known as the *smoothing transformation*.

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1 Introduction

1.1 Introduction and Background

Branching random walk (BRW) was introduced by Hammersley [22] in the early 1970s. Over the last five decades, it has received a lot of attention from various researchers in probability theory and statistical physics. The model, as such, is very simple to describe. It starts with one particle at the origin. After a unit amount of time, the particle dies and gives birth to a number of similar particles, which are placed at possibly different locations on the real line $\mathbb{R}$. These particles at possibly different places on $\mathbb{R}$ form the so-called first generation of the process and can be described through a point process, say Z on $\mathbb{R}$. After another unit time, each of the particles in the first generation behaves independently and identically as that of the parent, that is it dies, but before that, it produces a bunch of offspring particles which are displaced by independent copies of Z. The particles in generation one behave independently but identically of one another. The process then continues in the next unit of time and so on. The dynamics so produced is called a *branching random walk (BRW)*.

Let $N := Z(\mathbb{R})$ be the offspring distribution of the underlying branching process. As will be clear in the sequel (see Sect. 1.3), without loss of any generality throughout this article we will assume that $\mathbb{P}(N \geq 1) = 1$. As otherwise, all of our results will hold when the process is supercritical and we conditioned on its survival.

Let R_n denote the position of the *right-most particle* in the generation n. In the seminal works, Hammersley [22], Kingman [24], and Biggins [10] proved that under very minimal condition of the displacement point process Z,

$$\frac{R_n}{n} \longrightarrow \gamma \ \text{ a.s.,} \tag{1.1}$$

where $\gamma > 0$ is a constant associated with the displacement point process Z. It is worth mentioning here that if we forget about the position of the particles and only keep count of the number of particles, then it forms a Galton–Watson branching process with progeny distribution given by $Z(\mathbb{R})$. As noted in Aldous and Bandyopadhyay [4], the arguments of Hammersley [22] can be used to claim that if median (R_{n+1}) − median (R_n) remains bounded above, then the sequence of random variables $(R_n - \text{median}\,(R_n))_{n\geq 0}$ remains tight. Similar arguments also appear in Dekking and Host [20].

From historical point of view, it is interesting to note here that Biggins [10] wrote:

> "Of course pride of place in the open problems goes to establishing more detailed results than (1.1) of the kinds that are already available for branching Brownian motion."

Indeed, McKean [31] showed that for similar continuous time version with *Branching Brownian Motion (BBM)*, the maximum position, when centered by its median, converges weakly to a traveling wave solution. Later Bramson [14, 16] gave detailed order of the centering and showed that an "extra" logarithmic term appears, which later was termed as the *Bramson correction*. Later Lalley and Sellke [26] gave a different probabilistic interpretation of the traveling wave limit through certain conditional limit theorem and using a new concept called the *derivative martingales*.

In a series of papers, Bramson and Zeitouni [15, 17] showed that under fairly general conditions, $(R_n - \text{median}\,(R_n))_{n\geq 0}$ remains tight. And in 2009, two groups of researchers, Hu and Shi [23] and Addario-Berry and Reed [1], independently proved that $\frac{R_n}{n}$ has a second-order fluctuation which was identified as $-\frac{3}{2}\log n$ in probability. Finally, in 2013, Aïdékon [2] proved that $R_n - \gamma n + \frac{3}{2}\log n$ converges in law to a randomly shifted Gumbel distribution, essentially settling the long-standing open problem of Biggins [10]. We refer to [33] for an excellent review of the classical and recent results on BRW.

In recent days more generally, it is expected that this behavior for the maximum is shared by the universality class of what is known as the "*log-correlated fields*." We refer to [6] for a detailed review of such generalization and results there in.

In this work, we consider a modified version of the classical BRW. The modification is done at the last generation where we add *i.i.d.* displacements of a specific form. Since the modifications have been done only at the last generation, so we call this model *last progeny modified branching random walk* or abbreviate it as *LPM-BRW*. The model is described in more detail in the following subsection. We establish several results similar to Aïdékon [2] for our model and show that the limit has the desired universality. Further work on large deviation for the same model and centered limits for a similar but inhomogeneous displacements can be found in [21] and [7], respectively.

While we were preparing this manuscript Maillard and Mallein [29] considered a general framework for characterizing the limiting distribution of what they called "branching-type structure" via a fixed point of an operator referred to as the *branching convolution* introduced by Bertoin and Mallein [9] on the set of all point processes endowed with an appropriate topology. They mentions in their paper that our model is an example of their general framework (see fifth bullet point on page 2 of [29]). It is worth nothing here that [29] does not provide any general proof of convergence after appropriate centering but gives characterization of the limit given convergence. Our detailed analysis in this work provides a set of non-trivial and concrete cases where the result of [29] may be applied for characterization of the limit.

1.2 Model

Let $Z = \sum_{j\geq 1} \delta_{\xi_j}$ be a point process on $\mathbb{R}$ and $N := Z(\mathbb{R}) < \infty$ a.s. At the 0-th generation, we start with an initial particle at the origin. At time $n \geq 1$, each of the

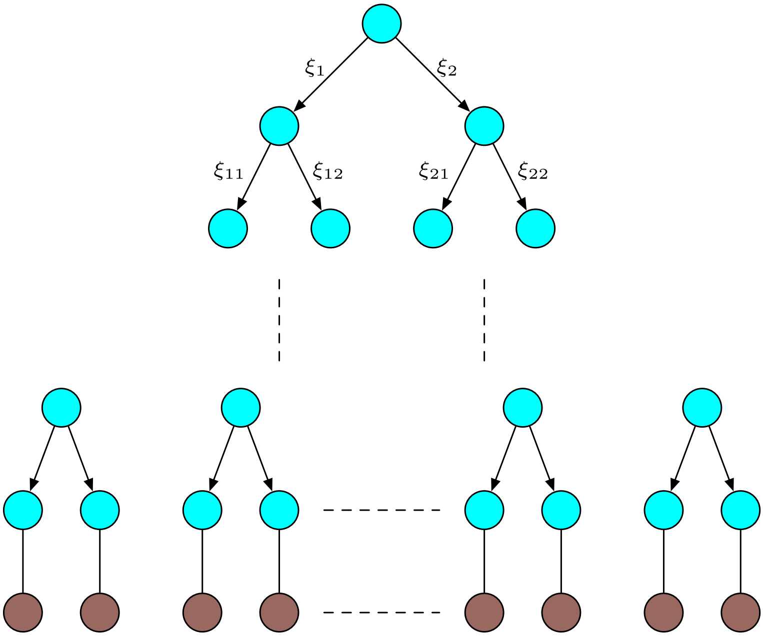

Fig. 1 Last progeny modified branching random walk (LPM-BRW)

particles at generation $(n-1)$ gives birth to a random number of offspring distributed according to N. The offsprings are then given random displacements independently and according to a copy of the point process Z.

For a particle v we shall denote its generation by $|v|$, i.e., $|v| = n$ if v belongs to the n-th generation. Let $S(v)$ denote the position of the particle v, which is the sum of all the displacements the particle v and its ancestors have received. The stochastic process $\left\{ S(v) \,\middle|\, |v| = n \right\}_{n \geq 0}$ is typically referred to as the classical *branching random walk (BRW)*. The quantity of interest is the maximum position, typically denoted by $R_n := \max_{|v|=n} S(v)$, is also the right-most position as discussed above.

In our model, we introduce two parameters. One is a positive real number, which we denote by $\theta > 0$. The other one is a positively supported distribution, which we will denote by $\mu \in \mathcal{P}(\bar{\mathbb{R}}_+)$. The parameter θ should be thought of as a *scaling parameter* for the extra displacement we give to each individual at the n-th generation. This extra displacement is as follows. At a generation $n \geq 1$, we give additional displacements to each of the particles at the generation n, which are of the form $\frac{1}{\theta} X_v := \frac{1}{\theta} (\log Y_v - \log E_v)$, where $\{Y_v\}_{|v|=n}$ are i.i.d. μ, while $\{E_v\}_{|v|=n}$ are i.i.d. Exponential (1), and they are independent of each other and also of the process $(S(u))_{|u| \leq n}$. We denote by $R_n^*(\theta, \mu)$ the maximum position of this *last progeny modified branching random walk (LPM-BRW)*. If the parameters θ and μ are clear from the context, then we will simply write this as R_n^*. A schematic of the process is given below.

1.3 Assumptions

Before we state our assumptions, we introduce the following important quantities. For a point process $Z = \sum_{j=1}^{N} \delta_{\xi_j}$, we will write

$$m(\theta) := \mathbb{E}\left[\int_{\mathbb{R}} e^{\theta x} Z(\mathrm{d}x)\right] = \mathbb{E}\left[\sum_{j=1}^{N} e^{\theta \xi_j}\right],$$

where $\theta \in \mathbb{R}$, whenever the expectation exists. Naturally, m is the *moment generating function* of the point process Z. Further, we define $\nu(t) := \log m(t)$ for $t \in \mathbb{R}$, whenever $m(t)$ is defined.

We now state our main assumptions. Throughout this paper, we will assume the following three conditions hold:

- **(A1)** $m(\theta) < \infty$ for all $\theta \in (-\vartheta, \infty)$ for some $\vartheta > 0$.
- **(A2)** The point process Z is *non-trivial*, and the *extinction probability* of the underlying *branching process* is 0. In other words, $\mathbb{P}(N = 1) < 1, \mathbb{P}(Z(\{t\}) = N) < 1$ for any $t \in \mathbb{R}$, and $\mathbb{P}(N \geq 1) = 1$.
- **(A3)** N has finite $(1 + p)$-th moment for some $p > 0$.

Remark 1.1 **(A1)** implies that m is infinitely differentiable on $(-\vartheta, \infty)$. Together with **(A3)**, it also implies that there exists $q > 0$, such that, for all $\theta \in [0, \infty)$,

$$\mathbb{E}\left[\left(\int_{\mathbb{R}} e^{\theta x}\, Z(\mathrm{d}x)\right)^{1+q}\right] < \infty. \tag{1.2}$$

Proof of this is given in the appendix (see Proposition A.1).

Notice also that under Assumptions **(A1)** and **(A2)**, $\nu(t)$ is strictly convex in $(-\vartheta, \infty)$. Though this is a well-known fact, we are unable to find an exact reference for this. So a proof of this has been given in the Appendix as Proposition A.2.

1.4 Motivation

Our main motivation to study this new LPM-BRW model is what we will see in the sequel that there is a nice *coupling* of R_n^* with a *linear statistic*, which is an additive martingale associated with BRW (see Corollary 3.6 for details). For such statistics, asymptotics can be computed using various martingale techniques, some of which are known. This novel connection is indeed the reason the model intrigued us. As illustrated in this article, our model is one example where this coupling technique works. This connection is novel and we believe that it has potential of many more applications.

The other motivation, and perhaps more straightforward one, is to be able to compare our results with the existing ones in the context of the classical BRW (such as, asymptotics derived in [2]). We see a difference appears in the constant factor in front of the Bramson correction (see Theorem 2.2), but the final weak limit remains the same. This in turn shows that the centered asymptotic results are heavily dependent on the displacements given at the end nodes, but not the limit. While doing this comparison, we also have been able to get the exact constant for the centered limit which was earlier not known (see Remark 2.6 for the details).

1.5 Outline

In Sect. 2, we state the main results. Section 3 provides our main tool: the coupling between the maximum statistic and a linear statistic. In Sect. 4, we state and prove a few asymptotic results about the associated *linear statistic*, which we later use in the proofs of the main results. We end with Sect. 5, where we give all the details of the proofs. For the sake of completeness, proofs of a few elementary results are provided in the Appendix.

2 Main Results

We start by defining a constant related to the underlying driving point process Z, which we denote by θ_0. Let

$$\theta_0 := \inf\left\{\theta > 0 : \frac{\nu(\theta)}{\theta} = \nu'(\theta)\right\}.$$

The fact that $\nu(\theta)$ is strictly convex ensures that the above set is at most singleton. If it is a singleton, then as illustrated in Fig. 2, θ_0 is the unique point in $(0, \infty)$ such that a tangent line from the origin to the graph of $\nu(\theta)$ touches the graph at $\theta = \theta_0$. And if it is empty, then by definition θ_0 takes value ∞, and there does not exist any tangent line from the origin to the graph of $\nu(\theta)$ on the right half-plane.

Remark 2.1 It is worth noting that $\nu(\theta)/\theta$ is strictly decreasing for $\theta \in (0, \theta_0)$ and strictly increasing for $\theta \in (\theta_0, \infty)$. Therefore, as shown in Fig. 3, when θ_0 is finite, it is the unique point of minimum for $\nu(\theta)/\theta$.

Remark 2.2 Note that

$$\frac{\nu(\theta)}{\theta} = \lim_{n\to\infty} \frac{1}{n\theta} \log \mathbb{E}\left[W_n(\theta)\right],$$

where $W_n(\theta) = W_n(\theta, 0)$ is as defined in (4.2). The quantity $\nu(\theta)/\theta$ is often referred to as the *"annealed free energy."* The so-called *quenched free energy*, denoted by $F(\theta)$, can be defined as

$$F(\theta) := \lim_{n\to\infty} \frac{1}{n\theta} \mathbb{E}\left[\log W_n(\theta)\right].$$

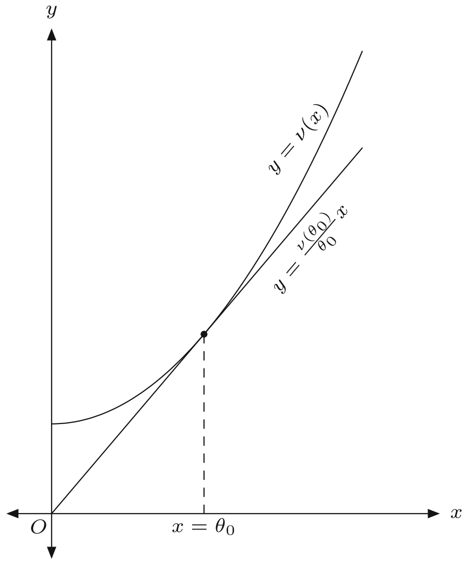

Fig. 2 Illustration of θ_0

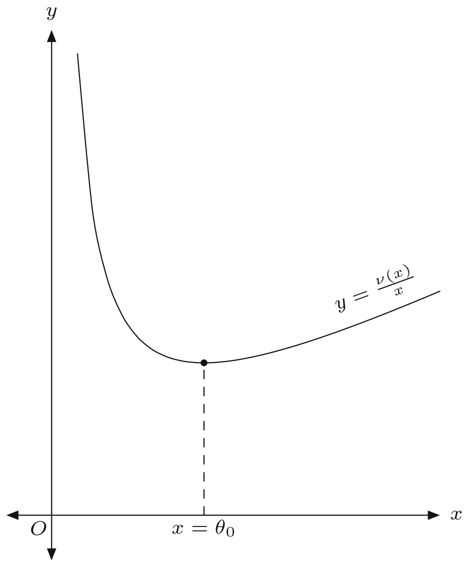

Fig. 3 Graph of $y = \nu(x)/x$

Using Jensen's inequality, it is easy to see that they satisfy the inequality

$$F(\theta) \leq \frac{\nu(\theta)}{\theta}.$$

Whether θ_0 is finite or infinite can be characterized by the fact that $\theta_0 < \infty$, if and only if,

$$\lim_{\theta\to\infty}\left(\nu(\theta)-\theta\left(\lim_{x\to\infty}\nu'(x)\right)\right)<0.$$

In the sequel we will see that θ_0 will be a point of *phase transition* for our process. Indeed, it may be viewed as the *critical inverse temperature* of the model, as it minimizes the limiting “*free energy*" (see Remark 2.2). We thus classify our model into three different classes depending on the parameter θ is *below*, *equal*, or *above* the quantity θ_0. We term these as *below the boundary case (BBC)*, *the boundary case (BC)*, and *above the boundary case (ABC)*, respectively, rather than *sub-critical*, *critical*, and *super-critical*. We adopt to this terminology following Biggins and Kyprianou [13] because our $\theta=\theta_0$ corresponds to what they call the *boundary case*.

2.1 Almost Sure Asymptotic Limit

Our first result is a *strong law of large number*-type result, which is similar to (1.1).

Theorem 2.1 *For every non-negatively supported probability $\mu\neq\delta_0$ that admits a finite mean, almost surely*

$$\frac{R_n^*(\theta,\mu)}{n}\to\begin{cases}\frac{\nu(\theta)}{\theta} & \text{if } \theta<\theta_0\leq\infty;\\ \frac{\nu(\theta_0)}{\theta_0} & \text{if } \theta_0\leq\theta<\infty.\end{cases}\tag{2.1}$$

Remark 2.3 Note that the almost sure limit remains the same as $\frac{\nu(\theta_0)}{\theta_0}$ for both the BC and the ABC.

2.2 Centered Asymptotic Limits

The centered asymptotic limits vary in the three different cases depending on the value of the parameter θ as described above. We thus state the results separately for the three cases.

2.2.1 The Boundary Case ($\theta=\theta_0<\infty$)

Theorem 2.2 *Assume that μ admits a finite mean, then there exists a random variable $H_{\theta_0}^{\infty}$, which may depend on θ_0, such that*

$$R_n^*-\frac{\nu\left(\theta_0\right)}{\theta_0}n+\frac{1}{2\theta_0}\log n\ \Rightarrow\ H_{\theta_0}^{\infty}+\frac{1}{\theta_0}\log\langle\mu\rangle,\tag{2.2}$$

where $\langle\mu\rangle$ is the mean of μ.

Remark 2.4 Notice that the coefficient for the linear term, which is $\nu(\theta_0)/\theta_0$, is exactly the same as that of the centering of R_n, as proved by Aïdékon [2]. However, the coefficient for the logarithmic term is 1/3-rd of that of the centering of R_n, as shown by Aïdékon [2]. The limiting distribution is also similar to that obtained by Aïdékon [2], which is a randomly shifted *Gumbel distribution*.

In fact, as we will see from the proof of the above theorem, we also have the following result (see Sect. 5):

Theorem 2.3 *Assume that μ admits a finite mean. Let*

$$\hat{H}_{\theta_0}^{\infty} = \frac{1}{\theta_0}\left[\log D_{\theta_0}^{\infty} + \frac{1}{2}\log\left(\frac{2}{\pi\sigma^2}\right)\right], \tag{2.3}$$

where

$$D_{\theta_0}^{\infty} \xlongequal{a.s.} \lim_{n\to\infty} -\frac{1}{m(\theta_0)^n}\sum_{|v|=n}(\theta_0 S(v) - n\nu(\theta_0))\, e^{\theta_0 S(v)}, \tag{2.4}$$

$$\sigma^2 := \mathbb{E}\left[\frac{1}{m(\theta_0)}\sum_{|v|=1}(\theta_0 S(v) - \nu(\theta_0))^2\, e^{\theta_0 S(v)}\right]. \tag{2.5}$$

Then

$$R_n^* - \frac{\nu(\theta_0)}{\theta_0}n + \frac{1}{2\theta_0}\log n - \hat{H}_{\theta_0}^{\infty} \Rightarrow \frac{1}{\theta_0}\left[\log\langle\mu\rangle - \log E\right], \tag{2.6}$$

where $E \sim$ Exponential (1) and $\langle\mu\rangle$ is the mean of μ.

Remark 2.5 We note here that the $H_{\theta_0}^{\infty}$ in Theorem 2.2 has the same distribution as $\hat{H}_{\theta_0}^{\infty} - \frac{1}{\theta_0}\log E$, where $E \sim$ Exponential (1) and is independent of $\hat{H}_{\theta_0}^{\infty}$.

Remark 2.6 One advantage of the above result is that we have been able to identify the exact additive constant, which is $\frac{1}{2}\log\left(\frac{2}{\pi\sigma^2}\right)$, for the result in Eq. (2.6). As far as we know, this was not discovered in any of the earlier works.

Remark 2.7 It is worth mentioning here that $D_{\theta_0}^{\infty}$ is indeed the almost sure limit of a *derivative martingale* defined by

$$D_n := -\sum_{|v|=n}(\theta_0 S(v) - \nu(\theta_0)n)e^{\theta_0 S(v) - \nu(\theta_0)n}$$

The idea of the derivative martingale originates from Lalley and Sellke [26], and later it also appears in Biggins and Kyprianou [12] as well as in Aïdékon [2].$D_{\theta_0}^{\infty} > 0$ a.s. under our assumptions and is a solution to a *linear recursive distributional equation (RDE)* given by

$$\Delta \xlongequal{d} \sum_{|v|=1} e^{\theta_0 S(v) - \nu(\theta_0)}\Delta_v, \tag{2.7}$$

where the Δ_v's are independent copies of Δ.

2.2.2 Below the Boundary Case ($\theta < \theta_0 \le \infty$)

Theorem 2.4 *Assume that μ admits a finite mean, then for $\theta < \theta_0 \le \infty$, there exists a random variable H_θ^∞, which may depend on θ, such that*

$$R_n^* - \frac{\nu(\theta)}{\theta} n \Rightarrow H_\theta^\infty + \frac{1}{\theta} \log\langle\mu\rangle, \tag{2.8}$$

where $\langle\mu\rangle$ is the mean of μ.

Remark 2.8 We note that in this case the *logarithmic correction* disappears.

Once again, just like in the boundary case, here too we have the following result also:

Theorem 2.5 *Assume that μ admits a finite mean. Let*

$$\hat{H}_\theta^\infty = \frac{1}{\theta} \log D_\theta^\infty,$$

where

$$D_\theta^\infty \stackrel{a.s.}{=\!=} \lim_{n\to\infty} \frac{1}{m(\theta)^n} \sum_{|v|=n} e^{\theta S(v)}, \tag{2.9}$$

which is also the mean 1 *solution of the following* linear RDE

$$\Delta \stackrel{d}{=\!=} \sum_{|v|=1} e^{\theta S(v) - \nu(\theta)} \Delta_v, \tag{2.10}$$

where the Δ_v's are independent copies of Δ. Then

$$R_n^* - \frac{\nu(\theta)}{\theta} n - \hat{H}_\theta^\infty \Rightarrow \frac{1}{\theta} \left[\log\langle\mu\rangle - \log E\right], \tag{2.11}$$

where $E \sim$ Exponential (1) *and $\langle\mu\rangle$ is the mean of μ.*

Remark 2.9 It is to be noted that the random variable H_θ^∞ in Theorem 2.4 has the same distribution as $\hat{H}_\theta^\infty - \frac{1}{\theta}\log E$, where $E \sim$ Exponential (1) and is independent of $\hat{H}_\theta^\infty$.

Remark 2.10 Biggins and Kyprianou [13] showed that under our assumptions, the solutions to the linear RDE given in (2.10) are unique up to a scale factor whenever they exist. Therefore D_θ^∞ is indeed the unique solution to the linear RDE (2.10) with mean 1.

2.2.3 Above the Boundary Case ($\theta_0 < \theta < \infty$)

Theorem 2.6 *Suppose $\mu = \delta_1$ and Z is non-lattice, that is, $\mathbb{P}(Z(a\mathbb{Z} + b) = N) < 1$ for all $a > 0$ and $b \in \mathbb{R}$, then for $\theta_0 < \theta < \infty$, there exists a constant $c_\theta \in \mathbb{R}$, which may depend on θ, such that*

$$R_n^* - \frac{\nu(\theta_0)}{\theta_0} n + \frac{3}{2\theta_0} \log n \;\Rightarrow\; H_{\theta_0}^\infty + c_\theta, \tag{2.12}$$

where $H_{\theta_0}^\infty$ is as in Theorem 2.2.

Remark 2.11 We would like to point out here that for the ABC, we have been able to prove the centered limit only for $\mu = \delta_1$. For technical reasons which will be clear from the proof, the general case may give a different result. See Remark 4.2 for more detail.

2.3 Brunet–Derrida-Type Results

In this section, we present results of the type Brunet and Derrida [18] for convergence of the extremal point processes. Their conjecture for the classical BRW was proven by Madaule [28]. Here we present similar results for our LPM-BRW. It is to be noted that the convergence of the point processes mentioned here is under the vague convergence topology on the set of all counting measures on $\mathbb{R}$.

Following Madaule [28], we now introduce point processes formed by the particles of appropriately re-centered branching random walks. For any $\theta < \theta_0 \le \infty$, we consider

$$Z_n(\theta) = \sum_{|v|=n} \delta_{\{\theta S(v) - \log E_v - n\nu(\theta) - \log D_\theta^\infty\}}, \tag{2.13}$$

where D_θ^∞ is defined in Theorem 2.5. And for $\theta = \theta_0 < \infty$, we consider

$$Z_n(\theta_0) = \sum_{|v|=n} \delta_{\left\{\theta_0 S(v) - \log E_v - n\nu(\theta_0) + \frac{1}{2}\log n - \log D_{\theta_0}^\infty - \frac{1}{2}\log\left(\frac{2}{\pi\sigma^2}\right)\right\}}, \tag{2.14}$$

where $D_{\theta_0}^\infty$ and σ^2 are as in Theorem 2.3.

Our first result is the weak convergence of the point processes $(Z_n(\theta))_{n \ge 0}$.

Theorem 2.7 *For $\theta < \theta_0 \le \infty$ or $\theta = \theta_0 < \infty$,*

$$Z_n(\theta) \xrightarrow{d} \mathcal{Y},$$

where $\mathcal{Y}$ is a Poisson point process on $\mathbb{R}$ with intensity measure $e^{-x}\,\mathrm{d}x$.

Following is a slightly weaker version of the above theorem, which is essentially a point process convergence of the appropriately centered LPM-BRW model.

Theorem 2.8 *For $\theta < \theta_0 \leq \infty$,*

$$\sum_{|v|=n} \delta_{\{\theta S(v) - \log E_v - n\nu(\theta)\}} \xrightarrow{d} \sum_{j\geq 1} \delta_{\zeta_j + \log D_\theta^\infty},$$

and for $\theta = \theta_0 < \infty$,

$$\sum_{|v|=n} \delta_{\left\{\theta_0 S(v) - \log E_v - n\nu(\theta_0) + \frac{1}{2}\log n\right\}} \xrightarrow{d} \sum_{j\geq 1} \delta_{\zeta_j + \log D_{\theta_0}^\infty + \frac{1}{2}\log\left(\frac{2}{\pi\sigma^2}\right)},$$

where $\mathcal{Y} = \sum_{j\geq 1} \delta_{\zeta_j}$ is a Poisson point process on $\mathbb{R}$ with intensity measure $e^{-x}\,\mathrm{d}x$, which is independent of the BRW.

Now, we denote $\mathcal{Y}_{\max}$ as the right-most position of the point process $\mathcal{Y}$, and we write $\overline{\mathcal{Y}}$ as the point process $\mathcal{Y}$ seen from its right-most position, that is,

$$\overline{\mathcal{Y}} = \sum_{j\geq 1} \delta_{\zeta_j - \mathcal{Y}_{\max}}.$$

The following result is an immediate corollary of the above theorem, which confirms that the *Brunet–Derrida Conjecture* holds for our model when $\theta < \theta_0 \leq \infty$ or $\theta = \theta_0 < \infty$.

Theorem 2.9 *For $\theta < \theta_0 \leq \infty$ or $\theta = \theta_0 < \infty$,*

$$\sum_{|v|=n} \delta_{\{\theta S(v) - \log E_v - \theta R_n^*(\theta, \delta_1)\}} \xrightarrow{d} \overline{\mathcal{Y}}.$$

Remark 2.12 Madaule [28] showed the convergence of the centered point process, obtained in the classical setup, to a decorated Poisson point process. As defined in [28], a decorated Poisson point process can be described as follows: Let $\mathfrak{Z} = \sum_{i\geq 1} \delta_{\zeta_i}$ be a Poisson point process with intensity $\lambda e^{-\alpha x}\,\mathrm{d}x$, and let $\{\mathfrak{X}_i\}_{i\geq 1}$ be independent copies of a point process $\mathfrak{X}$, where $\mathfrak{X}_i = \sum_{j\geq 1} \delta_{\chi_{i,j}}$. Then, the point process $\mathcal{Q} = \sum_{i\geq 1}\sum_{j\geq 1} \delta_{\zeta_i + \chi_{i,j}}$ is called a decorated Poisson point process with decoration $\mathfrak{X}$. In Madaule's work [28] the distribution of the decoration was left undescribed, which was later described in Mallein [30]. It is worth noting that, in our case, the decoration disappears. This is due to the fact that for both BC and BBC,

$$\max_{|v|=n} \frac{e^{\theta S(v)}}{W_n(\theta)} \xrightarrow{P} 0, \tag{2.15}$$

as mentioned in (4.7) and (4.9). However, as noted in Remark 4.2, (2.15) does not hold for the ABC. This added complication is the main reason that the results for the ABC remain open.

Remark 2.13 The point process $\overline{\mathcal{Y}}$ can be described explicitly in the following way: Let $\mathcal{N} = \sum_{j\geq 1} \delta_{\mathfrak{z}_j}$ be a homogeneous Poisson point process on $\mathbb{R}_+$ with intensity 1 and $E \sim \text{Exponential}\,(1)$ be independent of $\mathcal{N}$. Then

$$\overline{\mathcal{Y}} \stackrel{d}{=} \delta_0 + \sum_{j\geq 1} \delta_{-\log(1+(\mathfrak{z}_j/E))}.$$

3 Coupling Between a Maximum and a Linear Statistic

We start by defining a few operators on the space of probabilities which will help us to state and prove the coupling. In the sequel, $\mathcal{P}(A)$ will mean the set of all probabilities on a measurable space $(A, \mathcal{A})$, $\bar{\mathbb{R}} = [-\infty, \infty]$, $\bar{\mathbb{R}}_+ = [0, \infty]$ and $\text{dist}(X)$ represents the distribution of a random variable X. Let us also recall that $Z = \sum_{j\geq 1} \delta_{\xi_j}$ denotes a point process on $\mathbb{R}$ and $N := Z(\mathbb{R}) < \infty$ a.s.

Definition 3.1 *(Maximum Operator)* The operator $M_Z : \mathcal{P}(\bar{\mathbb{R}}) \to \mathcal{P}(\bar{\mathbb{R}})$ defined by

$$M_Z(\eta) = \text{dist}\left(\max_j\{\xi_j + X_j\}\right),$$

where $\{X_j\}_{j\geq 1}$ are i.i.d. $\eta \in \mathcal{P}(\bar{\mathbb{R}})$ and are independent of Z, will be called the *Maximum Operator*.

Remark 3.1 Observe that $M_Z^n(\eta)$ is the distribution of the maximum of the positions of the particles after adding i.i.d. displacements from η to the particles at n-th generation:

$$M_Z^n(\eta) = \text{dist}\left(\max_{|v|=n}\{S(v) + X_v\}\right).$$

In particular, $R_n \sim M_Z^n(\delta_0)$ and $R_n^* \sim M_Z^n(\eta)$, where η is the distribution of $\frac{1}{\theta}\log(Y_v/E_v)$ for a particle v at generation n.

Definition 3.2 *(Linear Operator)* The operator $L_Z : \mathcal{P}(\bar{\mathbb{R}}_+) \to \mathcal{P}(\bar{\mathbb{R}}_+)$ defined by

$$L_Z(\mu) = \text{dist}\left(\sum_{j\geq 1} e^{\xi_j} Y_j\right),$$

where $\{Y_j\}_{j\geq 1}$ are i.i.d. $\mu \in \mathcal{P}(\bar{\mathbb{R}}_+)$ and are independent of Z, will be called the *Linear* or *Smoothing Operator*.

Remark 3.2 Observe that $L_Z^n(\mu)$ is the distribution of $\sum_{|v|=n} e^{S(v)} Y_v$.

Definition 3.3 *(Link Operator)* The operator $\mathcal{E} : \mathcal{P}(\bar{\mathbb{R}}_+) \to \mathcal{P}(\bar{\mathbb{R}})$ defined by

$$\mathcal{E}(\mu) = \text{dist}\left(\log\frac{Y}{E}\right),$$

where $E \sim$ Exponential (1) and $Y \sim \mu \in \mathcal{P}(\bar{\mathbb{R}}_+)$ and they are independent, will be called the *Link Operator*.

Definition 3.4 For $a \geq 0$ and $b \in \mathbb{R}$, the operator $\Xi_{a,b}$ on the set of all point processes is defined by

$$\Xi_{a,b}(\mathcal{Z}) = \sum_{j\geq 1} \delta_{a\zeta_j - b},$$

where $\mathcal{Z} = \sum_{j\geq 1} \delta_{\zeta_j}$. Sometimes we may denote $\Xi_{a,0}$ by Ξ_a for notational simplicity.

The following result is one of the most important observations, and it links the operators defined above. As an immediate corollary, we get a very useful coupling between the LPM-BRW and the linear statistic associated with the *linear operator*.

Theorem 3.5 (Transforming Relationship) *For all $n \geq 1$,*

$$M_Z^n \circ \mathcal{E} = \mathcal{E} \circ L_Z^n. \tag{3.1}$$

Proof We first note that it is enough to show that Eq. (3.1) holds for $n = 1$, as the general case then follows by a trivial induction. To this end, let $Z = \sum_{j\geq 1} \delta_{\xi_j}$, $\{E_j\}_{j\geq 1}$ are i.i.d. Exponential (1), $\{Y_j\}_{j\geq 1}$ are i.i.d. μ, and they are independent of each other. Now,

$$\begin{aligned} M_Z \circ \mathcal{E}(\mu) &= \operatorname{dist}\left(\max_j \left(\xi_j + \log \frac{Y_j}{E_j}\right)\right) \\ &= \operatorname{dist}\left(\max_j \left(\log \frac{e^{\xi_j} Y_j}{E_j}\right)\right) \\ &= \operatorname{dist}\left(-\log\left(\min_j \frac{E_j}{e^{\xi_j} Y_j}\right)\right) = \operatorname{dist}\left(-\log \frac{E_1}{\sum_{j\geq 1} e^{\xi_j} Y_j}\right) = \mathcal{E} \circ L_Z(\mu). \end{aligned} \tag{3.2}$$

Note that the second-to-last equality in (3.2) comes from the fact that, conditionally on Z and $\{Y_j\}_{j\geq 1}$, the random variables $\left\{\frac{E_j}{e^{\xi_j} Y_j}\right\}_{j\geq 1}$ are independent and the conditional distribution of $\frac{E_j}{e^{\xi_j} Y_j}$ is Exponential $\left(e^{\xi_j} Y_j\right)$. Thus, using standard properties of exponential distribution, we conclude that conditionally on Z and $\{Y_j\}_{j\geq 1}$, the distribution of $\min_j \frac{E_j}{e^{\xi_j} Y_j}$ is Exponential $\left(\sum_{j\geq 1} e^{\xi_j} Y_j\right)$. □

Corollary 3.6 *Let $\theta > 0$ and $\mu \in \mathcal{P}(\bar{\mathbb{R}}_+)$. Then for any $n \geq 1$,*

$$\theta R_n^*(\theta, \mu) \stackrel{d}{=} \log Y_n^{\mu}(\theta) - \log E, \tag{3.3}$$

where $Y_n^\mu(\theta) := \sum_{|v|=n} e^{\theta S(v)} Y_v$, $\{Y_v\}_{|v|=n}$ *are i.i.d.* μ, *and* $E \sim \text{Exponential}(1)$ *and is independent of* Y_n^μ. *In other words,*

$$\mathbb{P}\left(R_n^*(\theta, \mu) \le x\right) = \mathbb{E}\left[e^{-e^{\theta x} \sum_{|v|=n} e^{\theta S(v)} Y_v}\right]. \tag{3.4}$$

Proof Observe that

$$\begin{aligned} \operatorname{dist}\left(\theta R_n^*(\theta, \mu)\right) &= M^n_{\Xi_\theta(Z)} \circ \mathcal{E}(\mu) \\ &= \mathcal{E} \circ L^n_{\Xi_\theta(Z)}(\mu) = \operatorname{dist}\left(\log Y_n^\mu - \log E\right). \end{aligned}$$

□

4 A Few Auxiliary Results on the Linear Statistic

In this section, we provide a few convergence results related to the *linear operator*, L_Z^n, as defined in the previous section and associated *linear statistic*, which is defined in the sequel (see Eq. (4.2)).

We start by observing that if we consider the point process $\Xi_{\theta, \nu_Z(\theta)}(Z)$, then

$$\nu_{\Xi_{\theta,\nu_Z(\theta)}(Z)}(\alpha) = \log \mathbb{E}\left[\int_{\mathbb{R}} e^{\alpha\theta x - \alpha \nu_Z(\theta)}\, Z(\mathrm{d}x)\right] = \nu_Z(\alpha\theta) - \alpha\nu_Z(\theta).$$

Differentiating this with respect to α, we get

$$\nu'_{\Xi_{\theta,\nu_Z(\theta)}(Z)}(\alpha) = \theta \nu'_Z(\alpha\theta) - \nu_Z(\theta).$$

Now, taking $\alpha = 1$, we have $\nu_{\Xi_{\theta,\nu_Z(\theta)}(Z)}(1) = 0$, and

$$\nu'_{\Xi_{\theta,\nu_Z(\theta)}(Z)}(1) = \theta \nu'_Z(\theta) - \nu_Z(\theta) \begin{cases} > 0 & \text{if } \theta_0 < \theta < \infty; \\ = 0 & \text{if } \theta = \theta_0 < \infty; \\ < 0 & \text{if } \theta < \theta_0 \le \infty. \end{cases}$$

Therefore, using [27, Theorem 1.6], we have

$$L^n_{\Xi_{\theta,\nu_Z(\theta)}(Z)}(\mu) \xrightarrow{w} \begin{cases} \delta_0 & \text{if } \theta = \theta_0 < \infty; \\ \mu_\theta^\infty & \text{if } \theta < \theta_0 \le \infty, \end{cases} \tag{4.1}$$

where for all $\theta < \theta_0$, $\mu_\theta^\infty \neq \delta_0$ is a fixed point of $L_{\Xi_{\theta,\nu_Z(\theta)}(Z)}$ and has the same mean as μ. Since $\mu_\theta^\infty \neq \delta_0$ is a fixed point of $L_{\Xi_{\theta,\nu_Z(\theta)}(Z)}$, we also have $\mu_\theta^\infty(\{0\}) = 0$ for all $\theta < \theta_0$.

We now define the *linear statistic* associated with the linear operator L_Z^n.

$$W_n(a,b) := \sum_{|v|=n} e^{aS(v)-nb}. \tag{4.2}$$

To simplify the notations, sometimes we may write $W_n(a,0)$ as $W_n(a)$. From the definition of the operator L, we get that

$$L^n_{\Xi_{a,b}(Z)}(\delta_1) = \operatorname{dist}\left(W_n(a,b)\right).$$

Since $\{W_n(\theta, \nu_Z(\theta))\}_{n\geq 1}$ is a non-negative martingale, it converges a.s. Therefore (4.1) implies that almost surely,

$$W_n(\theta, \nu_Z(\theta)) \to \begin{cases} 0 & \text{if } \theta = \theta_0 < \infty; \\ D_\theta^\infty & \text{if } \theta < \theta_0 \leq \infty, \end{cases} \tag{4.3}$$

for some positive random variable D_θ^∞ with $\mathbb{E}[D_\theta^\infty] = 1$, and the distribution of D_θ^∞ is a solution to the linear RDE (2.10).

The following proposition provides convergence results of $W_n(a,b)$ for various values of a and b.

Proposition 4.1 *For any $a > 0$ and $b \in \mathbb{R}$, almost surely*

$$W_n(a,b) \to \begin{cases} 0 & \text{if } a < \theta_0,\ b > \nu(a); & (i) \\ D_a^\infty & \text{if } a < \theta_0,\ b = \nu(a); & (ii) \\ \infty & \text{if } a < \theta_0,\ b < \nu(a); & (iii) \\ 0 & \text{if } \theta_0 < \infty,\ a \geq \theta_0,\ b \geq a\nu(\theta_0)/\theta_0; & (iv) \\ \infty & \text{if } \theta_0 < \infty,\ a \geq \theta_0,\ b < a\nu(\theta_0)/\theta_0. & (v) \end{cases}$$

To prove this proposition, we use the following elementary result. We provide the proof for sake of completeness.

Lemma 4.2 *Let $f : [0,\infty) \to \mathbb{R}$ be a continuously differentiable convex function and $\mathbb{S}$ be a convex subset of $[0,\infty) \times \mathbb{R}$ satisfying*

- *$(x,y) \in \mathbb{S}$ for all $0 < x < x_0$ and $y > f(x)$ and*
- *$(x,y) \notin \mathbb{S}$ for all $0 < x < x_0$ and $y < f(x)$,*

for some $x_0 > 0$. Then

$$\mathbb{S} \subseteq \left\{(x,y) : y \geq T_{x_0}(x)\right\},$$

where $T_{x_0}(\cdot)$ denotes the tangent line to f at x_0.

Proof We define a function $g : [0,\infty) \to \bar{\mathbb{R}}$ as

$$g(x) = \inf\left\{y : (x,y) \in \mathbb{S}\right\}.$$

We first show that g is convex. Take any x_1, x_2 such that $g(x_1)$, $g(x_2) < \infty$. By definition of g, for every $\epsilon > 0$, there exist $y_1 < g(x_1)+\epsilon$ and $y_2 < g(x_2)+\epsilon$ such that $(x_1, y_1), (x_2, y_2) \in \mathbb{S}$. So for any $\alpha \in (0, 1)$, $(\alpha x_1 + (1-\alpha)x_2, \alpha y_1 + (1-\alpha)y_2) \in \mathbb{S}$. Therefore

$$g(\alpha x_1 + (1-\alpha)x_2) \leq \alpha y_1 + (1-\alpha)y_2 < \alpha g(x_1) + (1-\alpha)g(x_2) + \epsilon.$$

As $\epsilon > 0$ is arbitrary, we have

$$g(\alpha x_1 + (1-\alpha)x_2) \leq \alpha g(x_1) + (1-\alpha)g(x_2),$$

and this is true for all $\alpha \in (0, 1)$. Therefore g is convex.

Let $T_x(.)$ be the tangent line to f at x. Since f is continuously differentiable, T_x converges pointwise to T_{x_0} as $x \to x_0$. Note that $g = f$ in $(0, x_0)$. Therefore, for all $x \in (0, x_0)$, T_x is also the tangent line to g at x. Since g is convex, we have $g \geq T_x$ for all $x \in (0, x_0)$. Hence, $g \geq T_{x_0}$. This completes the proof. □

Proof of Proposition 4.1 *Proof of (i),(ii), and (iii).* Noting that

$$W_n(a, b) = W_n(a, \nu(a)) \cdot e^{n(\nu(a)-b)}$$

(i), (ii), and (iii) follows from (4.3).

Proof of (iv). For $a \geq \theta_0$, we have

$$\begin{aligned} W_n(a, b) = \sum_{|v|=n} e^{aS(v)-nb} &\leq \left(\sum_{|v|=n} e^{(aS(v)-nb)\theta_0/a} \right)^{a/\theta_0} \\ &= W_n\left(\theta_0, b\theta_0/a\right)^{a/\theta_0} \\ &= \left(W_n\left(\theta_0, \nu(\theta_0)\right) \cdot e^{n(\nu(\theta_0)-b\theta_0/a)} \right)^{a/\theta_0} \end{aligned}$$

Since $W_n(a, b)$ is non-negative, using (4.3), we get that for $a \geq \theta_0$ and $b\theta_0/a \geq \nu(\theta_0)$,

$$W_n(a, b) \to 0 \text{ a.s.}$$

Proof of (v). Using (i) and (iii), we know that there exists $\mathcal{N} \subset \Omega$ with $\mathbb{P}(\mathcal{N}) = 0$ such that for all $\omega \notin \mathcal{N}$ and $(a, b) \in [(0, \theta_0) \times \mathbb{R}] \cap \mathbb{Q}^2$,

$$W_n(a, b)(\omega) \to \begin{cases} 0 & \text{if } b > \nu(a); \\ \infty & \text{if } b < \nu(a). \end{cases}$$

For any $\omega \notin \mathcal{N}$ and any subsequence $\{n_k\}$, we define

$$\mathbb{S}\left(\{n_k\}, \omega\right) = \left\{ (c, d) : \limsup_{k\to\infty} W_{n_k}(c, d)(\omega) < \infty \right\}.$$

Now, suppose $(c_1, d_1), (c_2, d_2) \in \mathbb{S}(\{n_k\}, \omega)$. Then for any $\alpha \in (0, 1)$,

$$\begin{aligned}
&\limsup_{k\to\infty} W_{n_k}(\alpha c_1 + (1-\alpha)c_2, \alpha d_1 + (1-\alpha)d_2)(\omega) \\
&\quad = \limsup_{k\to\infty} \sum_{|v|=n_k} \exp\left(\alpha\left[c_1 S(v)(\omega) - n_k d_1\right] + (1-\alpha)\left[c_2 S(v)(\omega) - n_k d_2\right]\right) \\
&\quad \le \alpha \left[\limsup_{k\to\infty} \sum_{|v|=n_k} \exp\left(c_1 S(v)(\omega) - n_k d_1\right)\right] \\
&\qquad + (1-\alpha)\left[\limsup_{k\to\infty} \sum_{|v|=n_k} \exp\left(c_2 S(v)(\omega) - n_k d_2\right)\right] \\
&\quad = \alpha\left[\limsup_{k\to\infty} W_{n_k}(c_1, d_1)(\omega)\right] + (1-\alpha)\left[\limsup_{k\to\infty} W_{n_k}(c_2, d_2)(\omega)\right] < \infty.
\end{aligned}$$

Therefore $\mathbb{S}(\{n_k\}, \omega)$ is convex. As $\mathbb{Q}^2$ is dense in $\mathbb{R}^2$, the conditions in Lemma 4.2 hold for the convex function ν, the convex set $\mathbb{S}(\{n_k\}, \omega)$, and the point θ_0. Thus for any $a \ge \theta_0$ and any $b < a\nu(\theta_0)/\theta_0$, we have $(a, b) \notin \mathbb{S}(\{n_k\}, \omega)$, which implies

$$\limsup_{k\to\infty} W_{n_k}(a, b)(\omega) = \infty.$$

This holds for all subsequence $\{n_k\}$ and all $\omega \notin \mathcal{N}$. Hence for all $a \ge \theta_0$ and all $b < a\nu(\theta_0)/\theta_0$, we have

$$W_n(a, b) \to \infty \text{ a.s.}$$

□

We recall that $W_n(\theta) = W_n(\theta, 0)$. The following corollary is a simple consequence of Proposition 4.1.

Corollary 4.3 *Almost surely*

$$\frac{\log W_n(\theta)}{n\theta} \to \begin{cases} \frac{\nu(\theta)}{\theta} & \text{if } \theta < \theta_0 \le \infty; \\ \frac{\nu(\theta_0)}{\theta_0} & \text{if } \theta_0 \le \theta < \infty. \end{cases}$$

Remark 4.1 To understand why the limit in Corollary 4.3 becomes constant for $\theta \ge \theta_0$, let us consider

$$\mathfrak{F}(\theta) = \lim_{n\to\infty} \frac{\log W_n(\theta)}{n\theta}.$$

Notice that $[W_n(\theta)]^{1/\theta}$ is indeed the ℓ_θ-norm of the sequence $\{e^{S_v}\}_{|v|=n}$. Thus, it is non-increasing in θ. Therefore, $\mathfrak{F}(\theta)$ is also non-increasing in θ. Now by the Cauchy-

Schwarz inequality, we get that for any $\theta_1, \theta_2 > 0$,

$$(W_n(\theta_1 + \theta_2))^2 \leq W_n(2\theta_1) \cdot W_n(2\theta_2).$$

Since dyadic rational numbers are dense in the real numbers, this gives us that for any $\alpha \in (0, 1)$,

$$W_n(\alpha\theta_1 + (1-\alpha)\theta_2) \leq W_n(\theta_1)^\alpha \cdot W_n(\theta_2)^{1-\alpha},$$

which means that $\log W_n(\theta)$ is convex in θ, and therefore so is $\theta\mathfrak{F}(\theta)$. Now, for $\theta < \theta_0$, $\mathfrak{F}(\theta) = \nu(\theta)/\theta$. So by Remark 2.1, the left derivative of $\mathfrak{F}$ is 0 at θ_0. Hence the right derivative is greater than or equal to 0 at θ_0, by convexity of the function $\theta \mapsto \theta\mathfrak{F}(\theta)$. Using again this convexity, it is now easy to show that $\mathfrak{F}'(\theta) \geq 0$ for all $\theta \geq \theta_0$; hence, $\mathfrak{F}(\theta) \geq \mathfrak{F}(\theta_0)$ for all $\theta \geq \theta_0$. But since $\mathfrak{F}$ is non-increasing, it has to be constant for $\theta \geq \theta_0$.

Proposition 4.4 *For $\theta < \theta_0 \leq \infty$ or $\theta = \theta_0 < \infty$,*

$$\frac{Y_n^\mu(\theta)}{W_n(\theta)} \xrightarrow{P} \langle\mu\rangle,$$

where $\langle\mu\rangle$ is the mean of μ and $Y_n^\mu(\theta)$ is as defined in Corollary 3.6.

Proof Recall that as in (4.3), for $\theta < \theta_0 \leq \infty$,

$$W_n(\theta, \nu(\theta)) \to D_\theta^\infty \text{ a.s.} \tag{4.4}$$

For $\theta_0 < \infty$, Aïdékon and Shi [3] have shown that under the assumptions in Sect. 1.3,

$$\sqrt{n}\, W_n(\theta_0, \nu(\theta_0)) \xrightarrow{P} \left(\frac{2}{\pi\sigma^2}\right)^{1/2} D_{\theta_0}^\infty, \tag{4.5}$$

where σ^2 and $D_{\theta_0}^\infty$ are as mentioned in Sect. 2.2.1. Also, Hu and Shi [23] have proved that under the assumptions in Sect. 1.3, for $\theta_0 < \theta < \infty$,

$$\frac{1}{\log n}\left(\log W_n(\theta) - \frac{\nu(\theta_0)}{\theta_0}\theta n\right) \xrightarrow{P} -\frac{3\theta}{2\theta_0}. \tag{4.6}$$

Now, observe that

$$\frac{Y_n^\mu(\theta)}{W_n(\theta)} - \langle\mu\rangle = \sum_{|v|=n}\left(\frac{e^{\theta S(v)}}{\sum_{|u|=n} e^{\theta S(u)}}\right)(Y_v - \langle\mu\rangle).$$

We define

$$M_n(\theta) := \max_{|v|=n} \frac{e^{\theta S(v)}}{\sum_{|u|=n} e^{\theta S(u)}} = \frac{e^{\theta R_n}}{W_n(\theta)}.$$

We recall that $W_n(a) = W_n(a, 0)$. For $\theta \in (0, \theta_0)$, we choose any $\theta_1 \in (\theta, \theta_0)$. Then we get

$$M_n(\theta) \le \frac{[W_n(\theta_1)]^{\theta/\theta_1}}{W_n(\theta)} \le \frac{[W_n(\theta_1, \nu(\theta_1))]^{\theta/\theta_1} \cdot e^{-n\theta\left(\frac{\nu(\theta)}{\theta} - \frac{\nu(\theta_1)}{\theta_1}\right)}}{W_n(\theta, \nu(\theta))}$$

Since ν is strictly convex, $\nu(\theta)/\theta$ is strictly decreasing for $\theta \in (0, \theta_0)$. Therefore using (4.4), we get

$$M_n(\theta) \to 0 \text{ a.s.} \tag{4.7}$$

For $\theta = \theta_0 < \infty$, we choose any $\theta_2 \in (\theta_0, \infty)$. Observe that

$$M_n(\theta_0) \le \frac{[W_n\,(\theta_2)]^{\theta_0/\theta_2}}{W_n(\theta_0)} = \frac{\left[n^{\theta_2/\theta_0} W_n\,(\theta_2, \theta_2\nu(\theta_0)/\theta_0)\right]^{\theta_0/\theta_2}}{n W_n(\theta_0, \nu(\theta_0))}. \tag{4.8}$$

Now, using (4.5), the denominator on the right-hand side of (4.8) goes to ∞ in probability, and by (4.6), the numerator goes to 0 in probability. Therefore, we obtain

$$M_n(\theta_0) \xrightarrow{P} 0. \tag{4.9}$$

Let $\mathcal{F}$ be the σ-field generated by the branching random walk, and $Y \sim \mu$. Then, using [11, Lemma 2.1], which is a particular case of [25, Lemma 2.2], we get that for every $0 < \varepsilon < 1/2$,

$$\begin{aligned} &\mathbb{P}\left(\left|\frac{Y_n^{\mu}(\theta)}{W_n(\theta)} - \langle\mu\rangle\right| > \varepsilon \,\middle|\, \mathcal{F}\right) \\ &\le \frac{2}{\varepsilon^2}\left(\int_0^{\frac{1}{M_n(\theta)}} M_n(\theta) t \cdot \mathbb{P}\left(|Y - \langle\mu\rangle| > t\right)\,\mathrm{d}t + \int_{\frac{1}{M_n(\theta)}}^{\infty} \mathbb{P}\left(|Y - \langle\mu\rangle| > t\right)\,\mathrm{d}t\right), \end{aligned}$$

which, by (4.7), (4.9), and dominated convergence theorem, converges to 0 in probability as $n \to \infty$. Then by taking expectation and using dominated convergence theorem again, we get

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\frac{Y_n^{\mu}(\theta)}{W_n(\theta)} - \langle\mu\rangle\right| > \varepsilon\right) = 0.$$

This completes the proof. □

Remark 4.2 We note here that Proposition 4.4 holds only when $\theta < \theta_0 \le \infty$ or $\theta = \theta_0 < \infty$. It is not clear that the conclusion of this proposition holds for the ABC, that is, when $\theta_0 < \theta < \infty$. In fact, in that case, $e^{\theta R_n} = \Theta_{\mathbb{P}}\left(W_n(\theta)\right)$ (follows from [2, Theorem 1.1] and the proof of Theorem 2.6 given below). Thus, (4.9) does not hold for the ABC.

5 Proofs of The Main Results

In this section we prove the main theorems. We start by proving the centered asymptotic limits: proving first Theorems 2.2 and 2.4 and then Theorems 2.3 and 2.5. Proof of Theorem 2.6 is given there after. We then prove the almost sure asymptotic limit, Theorem 2.1. Finally we end by proving the Brunet–Derrida-type results, Theorem 2.7 and Theorem 2.8.

5.1 Proof of Theorems 2.2 and 2.4

Proof Proposition 4.4, together with (4.4), gives us that for $\theta < \theta_0 \leq \infty$,

$$Y_n^{\mu}(\theta) \cdot e^{-n\nu(\theta)} \xrightarrow{P} D_\theta^\infty \cdot \langle \mu \rangle. \tag{5.1}$$

This implies

$$\log Y_n^{\mu}(\theta) - \log E - n\nu(\theta) \xrightarrow{P} \log D_\theta^\infty - \log E + \log\langle \mu \rangle, \tag{5.2}$$

where $E \sim$ Exponential (1) and is independent of $\{Y_v : |v| = n\}_{n\geq 0}$ and also independent of the BRW. Similarly, combining Proposition 4.4 and (4.5), we obtain that

$$Y_n^{\mu}(\theta_0) \cdot \sqrt{n} \cdot e^{-n\nu(\theta_0)} \xrightarrow{P} \left(\frac{2}{\pi\sigma^2}\right)^{1/2} \cdot D_{\theta_0}^\infty \cdot \langle \mu \rangle, \tag{5.3}$$

which implies

$$\log Y_n^{\mu}(\theta_0) - \log E - n\nu(\theta_0) + \frac{1}{2}\log n \xrightarrow{P} \frac{1}{2}\log\left(\frac{2}{\pi\sigma^2}\right) + \log D_{\theta_0}^\infty - \log E + \log\langle \mu \rangle. \tag{5.4}$$

Now, combining (5.2) and (5.4) together with Corollary 3.6 gives us the required result. □

5.2 Proof of Theorems 2.3 and 2.5

Proof By using a similar argument as in (3.2), we observe that

$$\begin{aligned}
\theta R_n^*(\theta, \mu) - \log Y_n^{\mu}(\theta) &= \max_{|v|=n} \left(\theta S(v) + \log Y_v - \log E_v\right) - \log\left(\sum_{|u|=n} e^{\theta S(u)} Y_u\right) \\
&= -\log\left(\min_{|v|=n} E_v \left(\frac{e^{\theta S(v)} Y_v}{\sum_{|u|=n} e^{\theta S(u)} Y_u}\right)^{-1}\right) \\
&\overset{d}{=} -\log E,
\end{aligned}$$

where $E \sim$ Exponential (1). Now, using Proposition 4.4, we obtain that for $\theta < \theta_0 \leq \infty$ or $\theta = \theta_0 < \infty$,

$$\theta R_n^*(\theta, \mu) - \log W_n(\theta) \;\Rightarrow\; \log\langle\mu\rangle - \log E.$$

This, together with (4.4) and (4.5), completes the proof. □

5.3 Proof of Theorem 2.6

Proof From [28, Theorem 2.3], it follows that under our assumptions, for $\theta_0 < \theta < \infty$, there exists a positive random variable $\mathfrak{D}_\theta$, which may depend on θ, such that

$$\log W_n(\theta) - \frac{\nu(\theta_0)}{\theta_0}\theta n + \frac{3\theta}{2\theta_0}\log n \;\Rightarrow\; \log \mathfrak{D}_\theta + \frac{\theta}{\theta_0}\log D_{\theta_0}^\infty, \tag{5.5}$$

where $\mathfrak{D}_\theta$ is independent of $D_{\theta_0}^\infty$. Since $W_n(\theta) = Y_n^{\delta_1}(\theta)$, using Corollary 3.6, we get that for $\theta_0 < \theta < \infty$,

$$R_n^*(\theta) - \frac{\nu\,(\theta_0)}{\theta_0}n + \frac{3}{2\theta_0}\log n \;\Rightarrow\; \frac{1}{\theta_0}\log D_{\theta_0}^\infty + \frac{1}{\theta}\log \mathfrak{D}_\theta - \frac{1}{\theta}\log E, \tag{5.6}$$

where $E \sim$ Exponential (1). We write the limiting random variable as H_θ^∞. Now, for u such that $|u| = 1$, we define

$$R_{n-1}^{*(u)}(\theta) := \left(\max_{v>u,|v|=n} S(v) - \frac{1}{\theta}\log E_v\right) - S(u).$$

Note that $\{R_{n-1}^{*(u)}(\theta)\}_{|u|=1}$ are i.i.d. and have the same distribution as $R_{n-1}^*(\theta)$. Now,

$$\begin{aligned} R_n^*(\theta) &= \max_{|u|=1}\left(\max_{v>u,|v|=n} S(v) - \frac{1}{\theta}\log E_v\right) \\ &= \max_{|u|=1}\left(S(u) + R_{n-1}^{*(u)}(\theta)\right). \end{aligned}$$

This implies

$$\begin{aligned} &R_n^*(\theta) - \frac{\nu\,(\theta_0)}{\theta_0}n + \frac{3}{2\theta_0}\log n \\ &\quad= \max_{|u|=1}\left(S(u) - \frac{\nu\,(\theta_0)}{\theta_0} + R_{n-1}^{*(u)}(\theta) - \frac{\nu\,(\theta_0)}{\theta_0}(n-1) + \frac{3}{2\theta_0}\log n\right). \end{aligned} \tag{5.7}$$

For $\theta_0 < \theta < \infty$, let $G_{\theta,n}$ be the distribution function of $R_n^*(\theta) - \frac{\nu(\theta_0)}{\theta_0} n + \frac{3}{2\theta_0} \log n$, and it converges pointwise to G_θ. Now, (5.7) tells us that

$$G_{\theta,n}(x) = \mathbb{E}\left[\prod_{|u|=1} G_{\theta,n-1}\left(x - S(u) + \frac{\nu(\theta_0)}{\theta_0} + \frac{3}{2\theta_0}\log\left(1 - \frac{1}{n}\right)\right)\right], \tag{5.8}$$

which implies

$$G_\theta(x) = \mathbb{E}\left[\prod_{|u|=1} G_\theta\left(x - S(u) + \frac{\nu(\theta_0)}{\theta_0}\right)\right] \tag{5.9}$$

If we define $g_\theta : (0, \infty) \to [0, 1]$ as $g_\theta(t) = G_\theta(-\log t)$, then from (5.9) we have

$$g_\theta(t) = \mathbb{E}\left[\prod_{|u|=1} g_\theta\left(t e^{S(u) - \frac{\nu(\theta_0)}{\theta_0}}\right)\right]. \tag{5.10}$$

Now, if $G_{\theta_0,n}$ is the distribution function of $R_n^*(\theta_0) - \frac{\nu(\theta_0)}{\theta_0} n + \frac{1}{2\theta_0} \log n$, and it converges pointwise to G_{θ_0}, then by defining $g_{\theta_0} : (0, \infty) \to [0, 1]$ as $g_{\theta_0}(t) = G_{\theta_0}(-\log t)$, a similar argument gives us

$$g_{\theta_0}(t) = \mathbb{E}\left[\prod_{|u|=1} g_{\theta_0}\left(t e^{S(u) - \frac{\nu(\theta_0)}{\theta_0}}\right)\right]. \tag{5.11}$$

Since both g_θ and g_{θ_0} are non-degenerate survival functions, (5.10) and (5.11), in conjunction with [5, Theorem 1.1], imply that $g_\theta(t) = g_{\theta_0}(t e^{c_\theta})$, for some $c_\theta \in \mathbb{R}$. Consequently, we get $G_\theta(x) = G_{\theta_0}(x - c_\theta)$, which means

$$H_\theta^\infty \stackrel{d}{=} H_{\theta_0}^\infty + c_\theta. \tag{5.12}$$

This completes the proof. $\square$

An alternative proof From [8, Theorem 1], we know

$$\mathbb{E}[e^{-t\mathfrak{D}_\theta}] = \begin{cases} e^{-(a_\theta t)^{\theta_0/\theta}} & \text{if } t \geq 0; \\ \infty & \text{if } t < 0, \end{cases} \tag{5.13}$$

for some $a_\theta > 0$. An alternative way to derive (5.12) from (5.6) is to show that

$$\frac{\mathfrak{D}_\theta}{E} \stackrel{d}{=} \frac{a_\theta}{E^{\theta/\theta_0}}. \tag{5.14}$$

Using an argument similar to that in Example 9.17 of [34], together with (5.13), we obtain that for any $x > 0$,

$$\mathbb{P}\left(\frac{a_\theta E}{\mathfrak{D}_\theta} > x\right) = \mathbb{E}\left[\mathbb{P}\left(E > \frac{x\mathfrak{D}_\theta}{a_\theta}\middle|\mathfrak{D}_\theta\right)\right] = \mathbb{E}\left[e^{-\frac{x\mathfrak{D}_\theta}{a_\theta}}\right] = e^{-x^{\theta_0/\theta}} = \mathbb{P}\left(E^{\theta/\theta_0} > x\right).$$

This proves (5.14), which implies (5.12). □

5.4 Proof of Theorem 2.1

Proof *(Upper bound).* Take any $\theta > 0$ and let $\beta = \min(\theta, \theta_0)$. Using Markov's inequality, we get that for every $\epsilon > 0$,

$$\mathbb{P}\left(\frac{R_n^*(\theta,\mu)}{n} - \frac{\nu(\beta)}{\beta} > \epsilon\right) \le e^{-n(\beta\epsilon+\nu(\beta))/2} \cdot \mathbb{E}\left[e^{\beta R_n^*(\theta,\mu)/2}\right].$$

Now, using Corollary 3.6, we have

$$\begin{aligned}
\mathbb{E}\left[e^{\beta R_n^*(\theta,\mu)/2}\right] &= \mathbb{E}\left[\left(\sum_{|v|=n} e^{\theta S(v)} Y_v\right)^{\beta/(2\theta)}\right] \cdot \mathbb{E}\left[E^{-\beta/(2\theta)}\right] \\
&\le \mathbb{E}\left[\sqrt{\sum_{|v|=n} e^{\beta S(v)} Y_v^{\beta/\theta}}\right] \cdot \Gamma\left(1 - \frac{\beta}{2\theta}\right) \\
&\le \sqrt{\mathbb{E}\left[\sum_{|v|=n} e^{\beta S(v)} Y_v^{\beta/\theta}\right]} \cdot \Gamma\left(1 - \frac{\beta}{2\theta}\right) \\
&= \sqrt{e^{n\nu(\beta)} \cdot \langle\mu\rangle_{\beta/\theta}} \cdot \Gamma\left(1 - \frac{\beta}{2\theta}\right),
\end{aligned}$$

where $\langle\mu\rangle_{\beta/\theta}$ is the (β/θ)-th moment of μ. So for every $\epsilon > 0$, we have

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\frac{R_n^*(\theta,\mu)}{n} - \frac{\nu(\beta)}{\beta} > \epsilon\right) < \infty. \tag{5.15}$$

Therefore using the Borel–Cantelli Lemma, we obtain for all $\theta > 0$, almost surely

$$\limsup_{n\to\infty} \frac{R_n^*(\theta,\mu)}{n} \le \begin{cases} \frac{\nu(\theta)}{\theta} & \text{if } \theta < \theta_0 \le \infty; \\ \frac{\nu(\theta_0)}{\theta_0} & \text{if } \theta_0 \le \theta < \infty. \end{cases} \tag{5.16}$$

(Lower bound). For u such that $|u| = m \leq n$, we define

$$R_{n-m}^{*(u)}(\theta, \mu) := \left(\max_{v > u, |v| = n} S(v) + \frac{1}{\theta} \log(Y_v / E_v) \right) - S(u).$$

Note that $\{R_{n-m}^{*(u)}(\theta, \mu)\}_{|u|=m}$ are i.i.d. and have the same distribution as $R_{n-m}^{*}(\theta, \mu)$. Now,

$$\begin{aligned} R_n^*(\theta, \mu) &= \max_{|u|=m} \left(\max_{v > u, |v| = n} S(v) + \frac{1}{\theta} \log(Y_v / E_v) \right) \\ &= \max_{|u|=m} \left(S(u) + R_{n-m}^{*(u)}(\theta, \mu) \right) \\ &\geq S(\tilde{u}_m) + \max_{|u|=m} \left(R_{n-m}^{*(u)}(\theta, \mu) \right), \end{aligned}$$

where

$$\tilde{u}_m := \arg\max_{|u|=m} \left(R_{n-m}^{*(u)}(\theta, \mu) \right).$$

Now, for any $\epsilon \in (0, 1)$ and for $\theta < \theta_0 \leq \infty$ or $\theta = \theta_0 < \infty$,

$$\begin{aligned} &\mathbb{P}\left(\frac{R_n^*(\theta, \mu)}{n} - \frac{\nu(\theta)}{\theta} < -\epsilon \right) \\ &\leq \mathbb{P}\left(S(\tilde{u}_{[\sqrt{n}]}) + \max_{|u|=[\sqrt{n}]} \left(R_{n-[\sqrt{n}]}^{*(u)}(\theta, \mu) \right) < n \left(\frac{\nu(\theta)}{\theta} - \epsilon \right) \right) \\ &\leq \mathbb{P}\left(\max_{|u|=[\sqrt{n}]} \left(R_{n-[\sqrt{n}]}^{*(u)}(\theta, \mu) \right) < n \left(\frac{\nu(\theta)}{\theta} - \frac{\epsilon}{2} \right) \right) \\ &\quad + \mathbb{P}\left(S(\tilde{u}_{[\sqrt{n}]}) < -\frac{n\epsilon}{2} \right) \\ &\leq \mathbb{E}\left[\mathbb{P}\left(R_{n-[\sqrt{n}]}^{*}(\theta, \mu) < n \left(\frac{\nu(\theta)}{\theta} - \frac{\epsilon}{2} \right) \right)^{N_{[\sqrt{n}]}} \right] + e^{-n\epsilon\vartheta/4} \cdot \mathbb{E}\left[e^{-\vartheta S(\tilde{u}_{[\sqrt{n}]})/2} \right]. \end{aligned}$$

Now, Corollary 4.3, together with Corollary 3.6 and Proposition (4.4), implies that for $\theta < \theta_0$ and also for $\theta = \theta_0 < \infty$,

$$\frac{R_n^*(\theta, \mu)}{n} \xrightarrow{p} \frac{\nu(\theta)}{\theta}.$$

Therefore for all large enough n,

$$\mathbb{P}\left(R_{n-[\sqrt{n}]}^{*}(\theta, \mu) < n \left(\frac{\nu(\theta)}{\theta} - \frac{\epsilon}{2} \right) \right) < \epsilon.$$

Observe, $N_{[\sqrt{n}]} < n$ implies at least $[\sqrt{n}] - \lceil \log_2 n \rceil$ many particles have given birth to only one offspring. Therefore

$$\mathbb{P}\left(N_{[\sqrt{n}]} < n\right) \leq (\mathbb{P}(N = 1))^{[\sqrt{n}] - \lceil \log_2 n \rceil}.$$

For the second term, we have

$$\mathbb{E}\left[e^{-\vartheta S(\tilde{u}_{[\sqrt{n}]})/2}\right] \leq \mathbb{E}\left[W_{[\sqrt{n}]}(-\vartheta/2)\right] = e^{[\sqrt{n}]\nu(-\vartheta/2)}.$$

Therefore we have for all large enough n,

$$\mathbb{P}\left(\frac{R_n^*(\theta,\mu)}{n} - \frac{\nu(\theta)}{\theta} < -\epsilon\right)$$
$$\leq \epsilon^n + (\mathbb{P}(N = 1))^{[\sqrt{n}] - \lceil \log_2 n \rceil} + e^{-n\epsilon\vartheta/4 + [\sqrt{n}]\nu(-\vartheta/2)}.$$

Since for every $\epsilon \in (0, 1)$,

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\frac{R_n^*(\theta,\mu)}{n} - \frac{\nu(\theta)}{\theta} < -\epsilon\right) < \infty, \tag{5.17}$$

using the Borel–Cantelli Lemma, we obtain that for $0 < \theta < \theta_0$ or $\theta = \theta_0 < \infty$,

$$\liminf_{n\to\infty} \frac{R_n^*(\theta,\mu)}{n} \geq \frac{\nu(\theta)}{\theta} \text{ a.s.} \tag{5.18}$$

To get an appropriate lower bound for $\theta_0 < \theta < \infty$, we need the following result; the proof of this is given at the end of this proof.

Proposition 5.1 *For any positively supported probability μ with finite mean, almost surely*

$$\frac{\log Y_n^{\mu}(\theta)}{n\theta} \to \begin{cases} \frac{\nu(\theta)}{\theta} & \textit{if } \theta < \theta_0 \leq \infty; \\ \frac{\nu(\theta_0)}{\theta_0} & \textit{if } \theta_0 \leq \theta < \infty. \end{cases}$$

Now observe that

$$\theta R_n^*(\theta,\mu) = \max_{|v|=n} (\theta S(v) + \log Y_v - \log E_v) \geq \max_{|v|=n} (\theta S(v) + \log Y_v) - \log E_{v_n},$$

where

$$v_n = \arg\max_{|v|=n} (\theta S(v) + \log Y_v).$$

Observe

$$Y_n^\mu(\theta+\theta_0) = \sum_{|v|=n} e^{(\theta+\theta_0)S(v)} Y_v \le W_n(\theta_0) \cdot e^{\max_{|v|=n}(\theta S(v)+\log Y_v)}.$$

Therefore we have

$$\frac{\theta R_n^*(\theta,\mu)}{n} \ge \frac{\log Y_n^\mu(\theta+\theta_0)}{n} - \frac{\log W_n(\theta_0)}{n} - \frac{\log E_{v_n}}{n}.$$

Since $\mathbb{E}[|\log E_{v_n}|]$ is finite, the Borel–Cantelli Lemma implies that the last terms on the right-hand side converge to 0 a.s. By Corollary 4.3 and Proposition 5.1, the first and the second terms a.s. converge to $(\theta+\theta_0)\nu(\theta_0)/\theta_0$ and $\nu(\theta_0)$, respectively. Thus whenever $\theta_0 < \infty$, we obtain that for all $\theta > \theta_0$,

$$\liminf_{n\to\infty} \frac{R_n^*(\theta,\mu)}{n} \ge \frac{\nu(\theta_0)}{\theta_0} \text{ a.s.} \tag{5.19}$$

This, together with (5.16) and (5.18), completes the proof. □

5.4.1 Proof of Proposition 5.1

Proof Corollary 3.6 says that

$$\theta R_n^*(\theta,\mu) \stackrel{d}{=} \log Y_n^\mu(\theta) - \log E.$$

Since $\mathbb{E}[|\log E|] < \infty$, (5.15) and (5.17), together with the Borel–Cantelli Lemma, imply that for $\theta < \theta_0 \le \infty$ and also for $\theta = \theta_0 < \infty$,

$$\frac{\log Y_n^\mu(\theta)}{n\theta} \to \frac{\nu(\theta)}{\theta} \text{ a.s.}$$

and for $\theta_0 < \theta < \infty$,

$$\limsup_{n\to\infty} \frac{\log Y_n^\mu(\theta)}{n\theta} \le \frac{\nu(\theta_0)}{\theta_0} \text{ a.s.}$$

So for any $a > 0$ and $b \in \mathbb{R}$, we have almost surely

$$Y_n^\mu(a,b) := Y_n^\mu(a)\cdot e^{-nb} \to \begin{cases} 0 & \text{if } a < \theta_0,\ b > \nu(a); \\ \infty & \text{if } a < \theta_0,\ b < \nu(a); \\ 0 & \text{if } \theta_0 < \infty,\ a \ge \theta_0,\ b > a\nu(\theta_0)/\theta_0. \end{cases}$$

Now, the exact similar argument as in the proof of Proposition 4.1(v) implies that for $\theta_0 < \infty$, $a \ge \theta_0$ and $b < a\nu(\theta_0)/\theta_0$,

$$Y_n^\mu(a,b) \to \infty \text{ a.s.}$$

Hence for $\theta_0 < \theta < \infty$,

$$\frac{\log Y_n^\mu(\theta)}{n\theta} \to \frac{\nu(\theta_0)}{\theta_0} \text{ a.s.}$$

This proves the proposition. □

5.5 Proof of Theorem 2.7

Proof Rényi's representation [32], together with the generalized version of it by Tikhov (see equation (3) of Tikhov [35]), gives us the following lemma.

Lemma 5.2 *Let* $\{E_{i,n} : 1 \le i \le m_n, n \ge 1\}$ *be an array of independent random variables with* $E_{i,n} \sim \text{Exponential}(\lambda_{i,n})$. *Suppose for all* $n \ge 1$, $\sum_{i=1}^{m_n} \lambda_{i,n} = 1$, *and* $\lim_{n\to\infty} \max_{i=1}^{m_n} \lambda_{i,n} = 0$. *Then as* $n \to \infty$, *the point process*

$$\sum_{i=1}^{m_n} \delta_{E_{i,n}} \xrightarrow{d} \mathcal{N},$$

where $\mathcal{N}$ *is a homogeneous Poisson point process on* $\mathbb{R}_+$ *with intensity* 1.

Now, let $\mathcal{F}$ be the σ-algebra generated by the branching random walk. We know that, conditionally on $\mathcal{F}$, $\{E_v W_n(\theta) e^{-\theta S(v)}\}$ are independent. Furthermore, conditionally on $\mathcal{F}$, $E_v W_n(\theta) e^{-\theta S(v)}$ follows Exponential $\left(\frac{e^{\theta S(v)}}{W_n(\theta)}\right)$. Note that

$$\sum_{|v|=n} \frac{e^{\theta S(v)}}{W_n(\theta)} = 1,$$

and by (4.7) and (4.9), we also have that for $\theta < \theta_0 \le \infty$ or $\theta = \theta_0 < \infty$,

$$\max_{|v|=n} \frac{e^{\theta S(v)}}{W_n(\theta)} \xrightarrow{P} 0.$$

Therefore by Lemma 5.2, for any positive integer k, Borel sets $B_1, B_2, \ldots, B_k$ and non-negative integers $t_1, t_2, \ldots, t_k$, we have

$$\mathbb{P}\left(\sum_{|v|=n} \delta_{E_v W_n(\theta) e^{-\theta S(v)}}(B_1) = t_1, \ldots, \sum_{|v|=n} \delta_{E_v W_n(\theta) e^{-\theta S(v)}}(B_k) = t_k \,\middle|\, \mathcal{F}\right)$$
$$\xrightarrow{P} \mathbb{P}\left(\mathcal{N}(B_1) = t_1, \ldots, \mathcal{N}(B_k) = t_k\right).$$

Then, using the dominated convergence theorem, we get

$$\mathbb{P}\left(\sum_{|v|=n} \delta_{E_v W_n(\theta) e^{-\theta S(v)}}(B_1) = t_1, \ldots, \sum_{|v|=n} \delta_{E_v W_n(\theta) e^{-\theta S(v)}}(B_k) = t_k\right) \\ \to \mathbb{P}\left(\mathcal{N}(B_1) = t_1, \ldots, \mathcal{N}(B_k) = t_k\right).$$

or equivalently (see Theorem 11.1.VII of Daley and Vere-Jones [19]),

$$\sum_{|v|=n} \delta_{E_v W_n(\theta) e^{-\theta S(v)}} \xrightarrow{d} \mathcal{N}. \tag{5.20}$$

Now for $\mathcal{N} = \sum_{j \geq 1} \delta_{\mathfrak{z}_j}$, we take $\mathcal{Y} = \sum_{j \geq 1} \delta_{-\log \mathfrak{z}_j}$. Clearly, $\mathcal{Y}$ is an inhomogeneous Poisson point process on $\mathbb{R}$ with intensity measure $e^{-x}\,\mathrm{d}x$. Since $-\log(.)$ is continuous and therefore Borel measurable, (5.20) implies that

$$\mathcal{U}_n := \sum_{|v|=n} \delta_{\theta S_v - \log E_v - \log W_n(\theta)} \xrightarrow{d} \mathcal{Y}. \tag{5.21}$$

To simplify the notations, for all $\theta < \theta_0 \leq \infty$, we denote

$$A_n(\theta) = n\nu(\theta) + \log D_\theta^\infty,$$

and for $\theta = \theta_0 < \infty$, we denote

$$A_n(\theta_0) = n\nu(\theta_0) - \frac{1}{2}\log n + \log D_{\theta_0}^\infty + \frac{1}{2}\log\left(\frac{2}{\pi\sigma^2}\right).$$

Recall that by (4.4) and (4.5), for $\theta < \theta_0 \leq \infty$ or $\theta = \theta_0 < \infty$,

$$A_n(\theta) - \log W_n(\theta) \xrightarrow{P} 0.$$

Now, take any positive integer k, non-negative integers $\{t_i\}_{i=1}^k$, and extended real numbers $\{a_i\}_{i=1}^k$ and $\{b_i\}_{i=1}^k$ with $a_i < b_i$ for all i. We choose $\delta \in \left(0, \min_{i=1}^k (b_i - a_i)/2\right)$. Then, we have

$$\begin{aligned}
&\mathbb{P}\left(\mathcal{U}_n\left((a_1 - \delta, b_1 + \delta)\right) \leq t_1, \ldots, \mathcal{U}_n\left((a_k - \delta, b_k + \delta)\right) \leq t_k\right) \\
&\quad - \mathbb{P}\left(|A_n(\theta) - \log W_n(\theta)| > \delta\right) \\
&\leq \mathbb{P}\left(Z_n(\theta)\left((a_1, b_1)\right) \leq t_1, \ldots, Z_n(\theta)\left((a_k, b_k)\right) \leq t_k\right) \\
&\leq \mathbb{P}\left(\mathcal{U}_n\left((a_1 + \delta, b_1 - \delta)\right) \leq t_1, \ldots, \mathcal{U}_n\left((a_k + \delta, b_k - \delta)\right) \leq t_k\right) \\
&\quad + \mathbb{P}\left(|A_n(\theta) - \log W_n(\theta)| > \delta\right).
\end{aligned}$$

Now, by (5.21), we have $\mathcal{U}_n \xrightarrow{d} \mathcal{Y}$. Since $\mathcal{Y}$ is a Poisson point process, it is continuous. Therefore, allowing $n \to \infty$ and then letting $\delta \to 0$, we obtain

$$\lim_{n\to\infty} \mathbb{P}\left(Z_n(\theta)\left((a_1, b_1)\right) \le t_1, \ldots, Z_n(\theta)\left((a_k, b_k)\right) \le t_k\right)$$
$$= \mathbb{P}\left(\mathcal{Y}\left((a_1, b_1)\right) \le t_1, \ldots, \mathcal{Y}\left((a_k, b_k)\right) \le t_k\right),$$

or equivalently, $Z_n(\theta) \xrightarrow{d} \mathcal{Y}$. This completes the proof. □

5.6 Proof of Theorem 2.8

Proof This is a slightly weaker version. It follows from arguments similar to those of the proof of Theorem 2.7. □

Appendix

Proposition A.1 *Under assumptions* **(A1)** *and* **(A3)**, *there exists* $q > 0$ *such that* (1.2) *holds.*

Proof Observe that

$$\int_{\mathbb{R}} e^{\theta x}\, Z(\mathrm{d}x) \le N e^{\theta\left(\max_{j=1}^N \xi_j\right)}, \quad \text{and} \quad e^{\theta\left(\max_{j=1}^N \xi_j\right)} \le \int_{\mathbb{R}} e^{\theta x}\, Z(\mathrm{d}x).$$

Now, using Hölder's inequality, we have

$$\begin{aligned}
\mathbb{E}\left[\left(\int_{\mathbb{R}} e^{\theta x}\, Z(\mathrm{d}x)\right)^{1+q}\right] &\le \mathbb{E}\left[N^{1+q} \cdot e^{\theta(1+q)\left(\max_{j=1}^N \xi_j\right)}\right] \\
&\le \left(\mathbb{E}\left[N^{(1+q)^2}\right]\right)^{\frac{1}{1+q}} \cdot \left(\mathbb{E}\left[e^{\theta\left(\frac{(1+q)^2}{q}\right)\left(\max_{j=1}^N \xi_j\right)}\right]\right)^{\frac{q}{1+q}} \\
&\le \left(\mathbb{E}\left[N^{(1+q)^2}\right]\right)^{\frac{1}{1+q}} \cdot \left(\mathbb{E}\left[\int_{\mathbb{R}} e^{\theta\left(\frac{(1+q)^2}{q}\right)x}\, Z(\mathrm{d}x)\right]\right)^{\frac{q}{1+q}} \\
&= \left(\mathbb{E}\left[N^{(1+q)^2}\right]\right)^{\frac{1}{1+q}} \cdot \left(m\left(\theta(1+q)^2/q\right)\right)^{\frac{q}{1+q}}.
\end{aligned}$$

Then, by choosing q such that $(1+q)^2 \le 1+p$, one gets (1.2). □

Proposition A.2 *Under assumptions* **(A1)** *and* **(A2)**, *the function* $\theta \mapsto \nu(\theta)$ *is strictly convex inside the open interval* $(-\vartheta, \infty)$.

Proof From Assumption **(A1)**, we know that

$$m(\theta) := \mathbb{E}\left[\int_{\mathbb{R}} e^{\theta x}\, Z(\mathrm{d}x)\right] < \infty,$$

for all $\theta \in (-\vartheta, \infty)$. Therefore using dominated convergence theorem, we have for all $\theta \in (-\vartheta, \infty)$,

$$m'(\theta) = \mathbb{E}\left[\int_{\mathbb{R}} x e^{\theta x}\, Z(\mathrm{d}x)\right] < \infty,$$

and

$$m''(\theta) = \mathbb{E}\left[\int_{\mathbb{R}} x^2 e^{\theta x}\, Z(\mathrm{d}x)\right] < \infty.$$

From Assumption **(A2)**, we have that $\mathbb{P}(Z(\{t\}) = N) < 1$ for all $t \in \mathbb{R}$. Therefore for all $t \in \mathbb{R}$,

$$\begin{aligned}
&\mathbb{E}\left[\int_{\mathbb{R}} (x-t)^2 e^{\theta x}\, Z(\mathrm{d}x)\right] > 0\\
&\quad\Rightarrow \mathbb{E}\left[\int_{\mathbb{R}} x^2 e^{\theta x}\, Z(\mathrm{d}x)\right] - 2t\mathbb{E}\left[\int_{\mathbb{R}} x e^{\theta x}\, Z(\mathrm{d}x)\right] + t^2\mathbb{E}\left[\int_{\mathbb{R}} e^{\theta x}\, Z(\mathrm{d}x)\right] > 0\\
&\quad\Rightarrow \mathbb{E}\left[\int_{\mathbb{R}} x^2 e^{\theta x}\, Z(\mathrm{d}x)\right] \cdot \mathbb{E}\left[\int_{\mathbb{R}} e^{\theta x}\, Z(\mathrm{d}x)\right] > \left(\mathbb{E}\left[\int_{\mathbb{R}} x e^{\theta x}\, Z(\mathrm{d}x)\right]\right)^2\\
&\quad\Rightarrow m''(\theta)m(\theta) > \left(m'(\theta)\right)^2.
\end{aligned}$$

Hence we have for all $\theta \in (-\vartheta, \infty)$,

$$\nu''(\theta) = \frac{m''(\theta)m(\theta) - \left(m'(\theta)\right)^2}{(m(\theta))^2} > 0.$$

This proves the proposition. □

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Declarations

Conflict of interest We declare that we have no conflict of interest as defined by Springer, the publisher, or other interests that might be perceived to influence the results and/or discussion reported in this paper.

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